Power Flow Analysis – Part 4

ELEN90060

Keith Moffat

Two Different AC Power Flow Algorithms

Newton-Raphson

$$-\begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}\begin{bmatrix} \Delta\theta^{(k)} \\ \Delta V^{(k)} \end{bmatrix} = \begin{bmatrix} f_P\left(\theta^{(k)}, V^{(k)}\right) \\ f_Q\left(\theta^{(k)}, V^{(k)}\right) \end{bmatrix}$$

$$\begin{bmatrix} \theta^{(k+1)} \\ V^{(k+1)} \end{bmatrix} = \begin{bmatrix} \theta^{(k)} \\ V^{(k)} \end{bmatrix} + \begin{bmatrix} \Delta\theta^{(k)} \\ \Delta V^{(k)} \end{bmatrix}$$

Fixed-Point Current-Injection

$$\boldsymbol{I}_P^{(k)} = \phi_P\left(\boldsymbol{V}_P^{(k)}\right)$$

$$\boldsymbol{Y}_{PP}\left(\boldsymbol{V}_P^{(k+1)} - 1\right) = \boldsymbol{I}_P^{(k)}$$

These ()s indicate function arguments

These ()s group the vector being multiplied

These are the two relevant AC Power Flow algorithms today.

Fixed-Point Current-Injection is an entirely different method than Newton-Raphson.

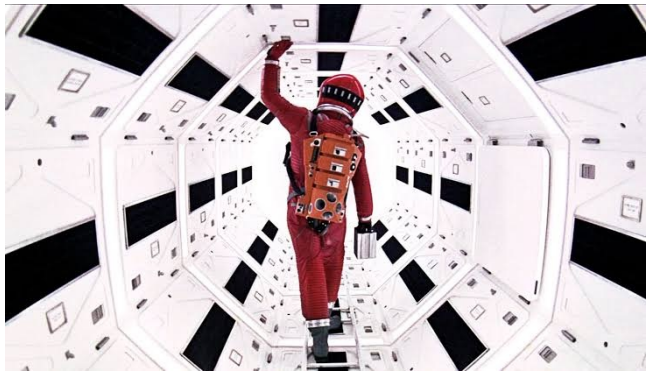
So this lecture will be an entirely different approach to the same AC Power Flow problem.

P: all non-slack buses. Q: PQ buses.

Newton–Raphson vectors: angles at all non-slack buses, magnitudes at PQ buses.

What is this?









The Datatron-204

Year: 1954

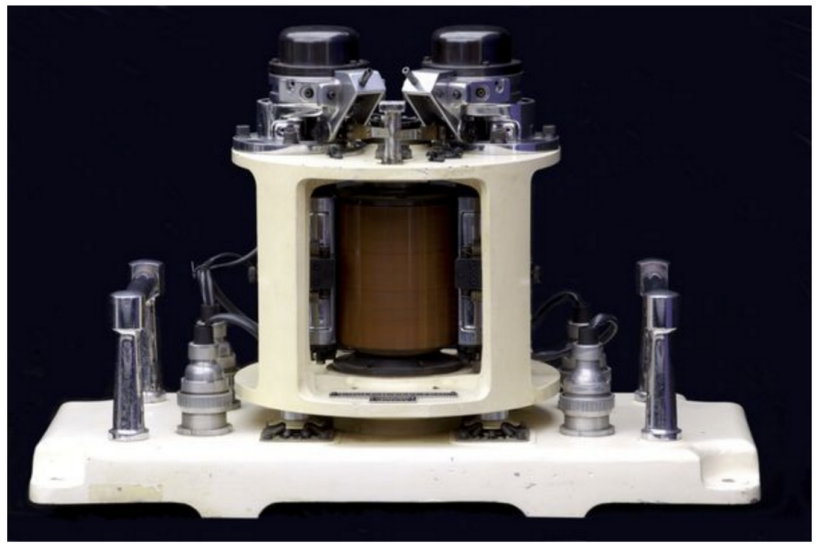
Magnetic Drum Memory

Deuce memory drum

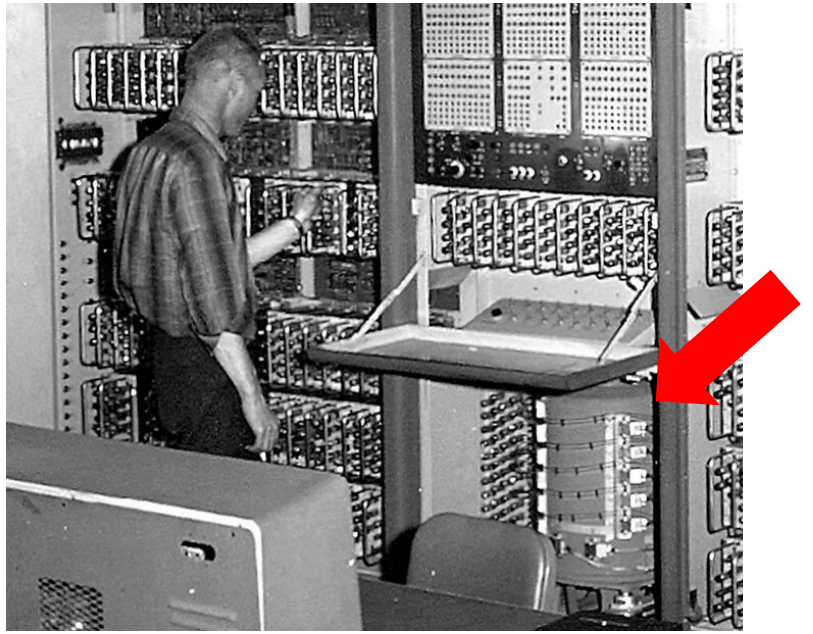
Appears In: **Magnetic Drums**

Details | **Description**

The Deuce computer was an early vacuum tube machine based on a design by British mathematician and WWII code breaker Alan Turing. This drum memory held 32 kB.

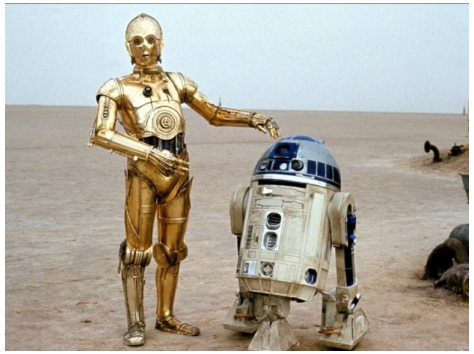


https://www.computerhistory.org/revolution/memory-storage/8/252/957



https://www.flickr.com/photos/usgeologicalsurvey/17165153732

R2-D2 version 0.0



Star Wars: George Lucas believes R2D2 was the one narrating sci-fi saga

Recall: History of Power Flow

Before Digital Computers: Physical Power Flow

- 1920s–1950s: engineers studied operating points using hand calculations and network analyzers.
- Network analyzers were physical scale models of the grid: miniature impedances, sources, and loads.

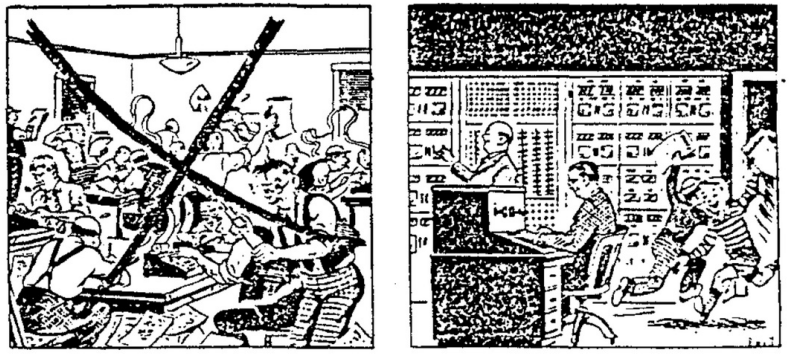
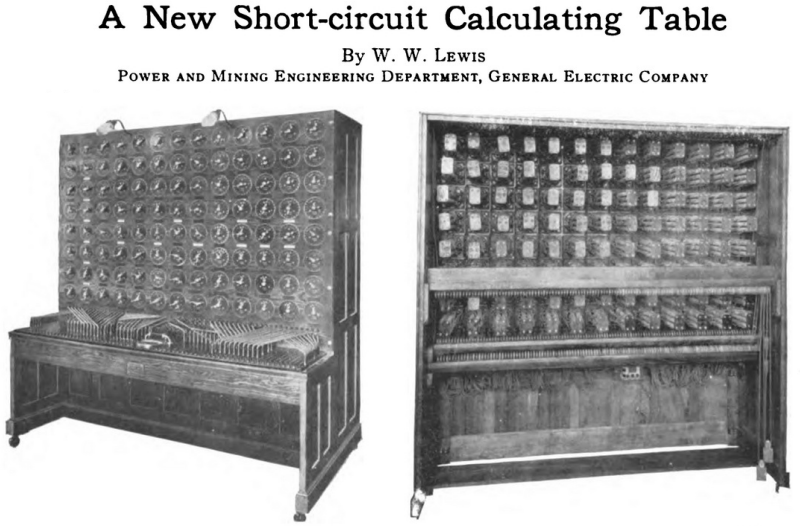
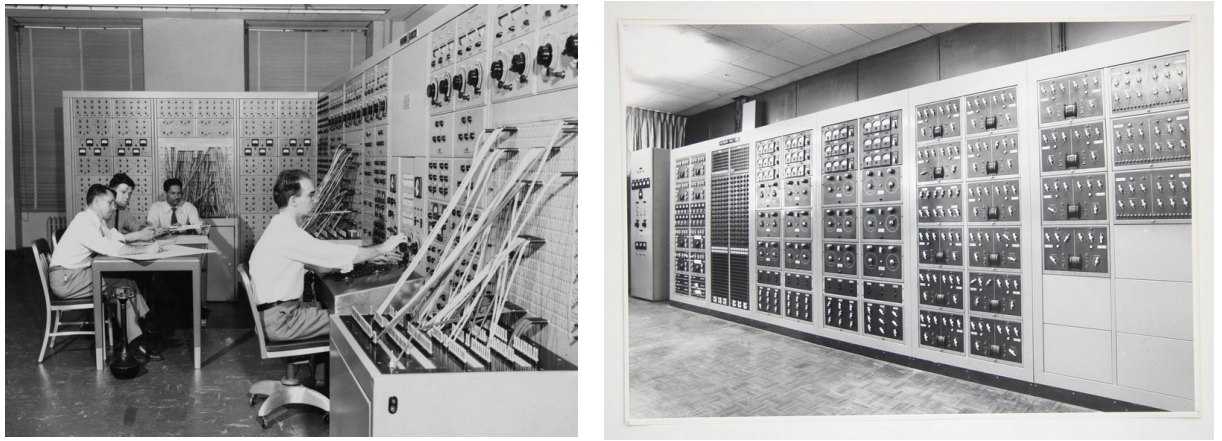


Fig. 1. Drawings of the calculating workshop (left) before and (right) after the introduction of the electrical analyzer.

Aristotle Tympas: From Digital to Analog https://ieeexplore.ieee.org/document/539915



W. W. Lewis, “A new short-circuit calculating table,” *General Electric Review*, pp. 669–671, 1920
https://babel.hathitrust.org/cgi/pt?id=nyp.33433062631878&seq=125



Photograph - The Westinghouse AC Network Analyser. Copyright Museums Victoria

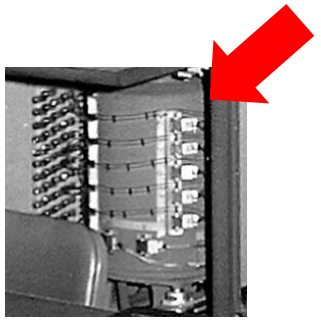
1956 first successful digital AC Power Flow

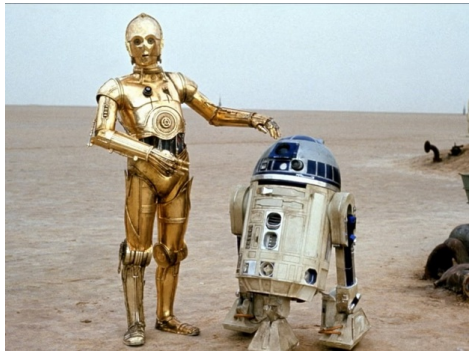
Digital Computer Solution of Power-Flow Problems

J. B. WARD MEMBER AIEE H. W. HALE ASSOCIATE MEMBER AIEE

Automatic Calculation of Load Flows

A. F. GLIMN MEMBER AIEE G. W. STAGG ASSOCIATE MEMBER AIEE





Gauss–Seidel reduced memory requirements by avoiding a linear solve and instead updating the voltage vector entries sequentially using:

$$\boldsymbol{V}_h^{(k+1)} = 1 + \frac{1}{\boldsymbol{Y}_{hh}}\left(\boldsymbol{I}_h^{(k)} - \sum_{i=1}^{h-1} \boldsymbol{Y}_{hi}\left(\boldsymbol{V}_i^{(k+1)} - 1\right) - \sum_{i=h+1}^{n} \boldsymbol{Y}_{hi}\left(\boldsymbol{V}_i^{(k)} - 1\right)\right)$$

simple network. This computation has been organized for and worked out on the Purdue University electrodata computer. This is a medium-speed stored-program computer with a 4,000-word magnetic drum memory. The same method can,



Yes, Ward and Hale really used a computer with blinking lights

1956 first successful digital AC Power Flow

Digital Computer Solution of Power-Flow Problems

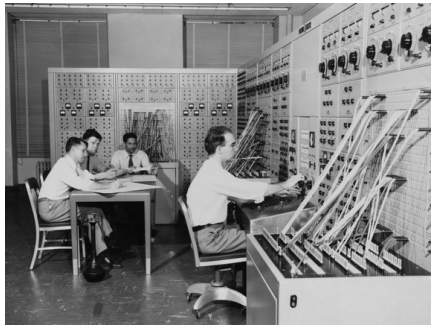
J. B. WARD
MEMBER AIEE

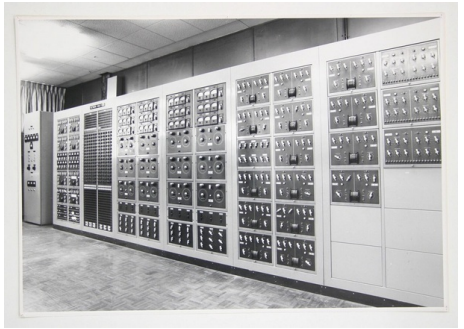
H. W. HALE
ASSOCIATE MEMBER AIEE

sults of adequate accuracy. In some studies, such as analysis of losses and incremental losses, the network analyzer does not provide sufficient precision and, in such cases, the digital computer solution gains a distinct advantage.



The Datatron-204





Photograph - The Westinghouse AC Network Analyser. Copyright Museums Victoria

Gauss-Seidel Power Flow Interleaves Current and Voltage Updates

Go through each bus sequentially.

At bus i, update the current from its present voltage: $\boldsymbol{I}_i^{(k)} = \overline{\left(\frac{\boldsymbol{S}_i}{\boldsymbol{V}_i^{(k)}}\right)}$

Immediately update that bus voltage using all new values already available:

This clever indexing allows the computer to update $\boldsymbol{V}$ without ever solving the vector-valued equation $\boldsymbol{YV} = \boldsymbol{I}$

$$\boldsymbol{V}_h^{(k+1)} = 1 + \frac{1}{\boldsymbol{Y}_{hh}}\left(\boldsymbol{I}_h^{(k)} - \sum_{i=1}^{h-1} \boldsymbol{Y}_{hi}\left(\boldsymbol{V}_i^{(k+1)} - 1\right) - \sum_{i=h+1}^{n} \boldsymbol{Y}_{hi}\left(\boldsymbol{V}_i^{(k)} - 1\right)\right)$$

$$\boxed{I_1^{(k)} \to V_1^{(k+1)}} \longrightarrow \boxed{I_2^{(k)} \to V_2^{(k+1)}} \longrightarrow \cdots \longrightarrow \boxed{I_n^{(k)} \to V_n^{(k+1)}} \longrightarrow \boxed{I_1^{(k+1)} \to V_1^{(k+2)}} \longrightarrow \boxed{I_2^{(k+1)} \to V_2^{(k+2)}} \longrightarrow \cdots$$

Update one current, immediately update its voltage, then move to the next bus.

Stop when the voltages converge: $\max \left|\boldsymbol{V}_i^{(k+1)} - \boldsymbol{V}_i^{(k)}\right| < \varepsilon$

History of Power Flow

Gauss Seidel was replaced by Newton-Raphson Power Flow

1956: Ward and Hale publish the first successful digital-computer solution of AC power-flow

1957: Glimn and Stagg introduce the Gauss-Seidel algorithm for AC power-flow

- Run on early drum-memory, vacuum-tube computers
- small networks (~50 nodes)

1967: Tinney and Hart: Newton's method + transistor computers + sparse matrix techniques

- Run on early transistor computers
- large (1000-node) AC power flow



Calculating the Grid: A Late 20th Century Story, Julie Cohn
https://ieeexplore.ieee.org/document/11475571



IBM 7040. Photo: Columbia photo archive

IEEE TRANSACTIONS ON POWER APPARATUS AND SYSTEMS VOL. PAS-86, NO. 11 NOVEMBER 1967

Power Flow Solution by Newton's Method

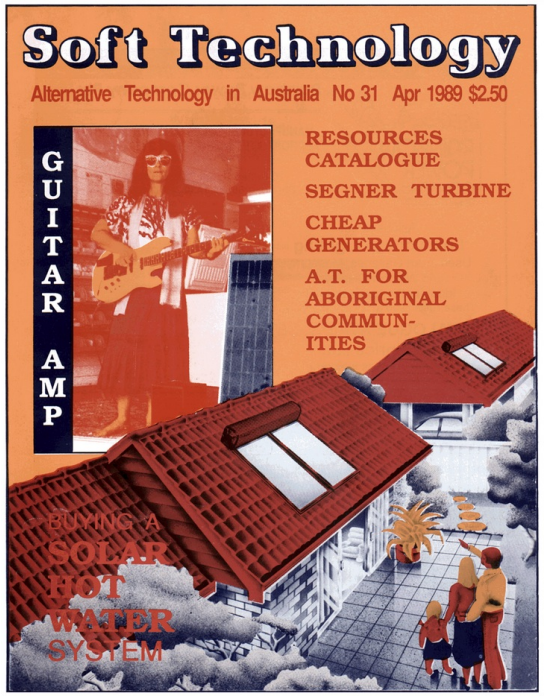
WILLIAM. F. TINNEY, SENIOR MEMBER, IEEE, AND CLIFFORD E. HART, MEMBER, IEEE

Fast-forward 30 years, weird people in Australia started attaching “solar panels” to the grid





https://renewportal.org.au/digital-library/





specifically, to the distribution grid

Actually, this was happening around the world

1973, Delaware, USA



https://www.udel.edu/udaily/2022/may/institute-energy-conversion-photovoltaics-solar-technology

1992, Sanyo Electric, Japan



https://www.solar-partners.jp/contents/35285.html

1994, UNSW, Australia



https://reneweconomy.com.au/why-first-grid-connected-solar-pv-didnt-destroy-the-grid-59290

1997, Florinopolis, Brazil



https://utfpr-ct-static-content.s3.amazonaws.com/labens.ct.utfpr.edu.br/wp-content/uploads/2021/11/Long-term-performance-of-the-first-grid-connected-building-integrated-amorphous-silicon-PV-installation-in-Brazil.pdf

1989, Chur, Switzerland





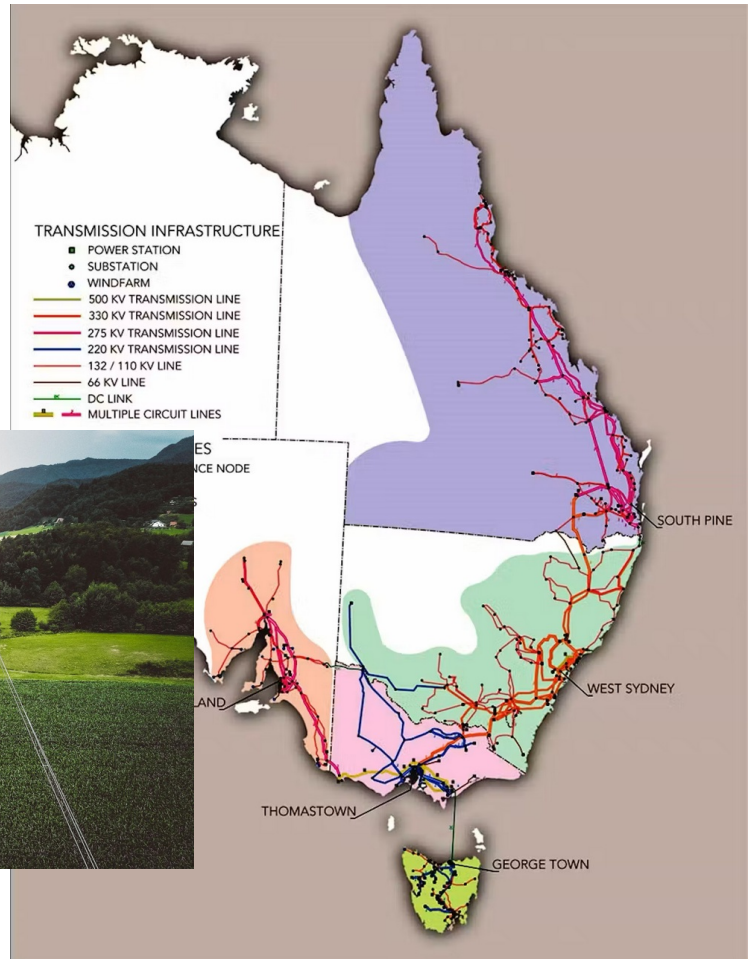
https://www.pvresources.com/en/pvpowerplants/noisebarriers.php?

Solar panels, and other distributed energy resources, are connected to the distribution grid.

Prior to the 1990s, grid operators largely ignored distribution grids, treating them as PQ loads.

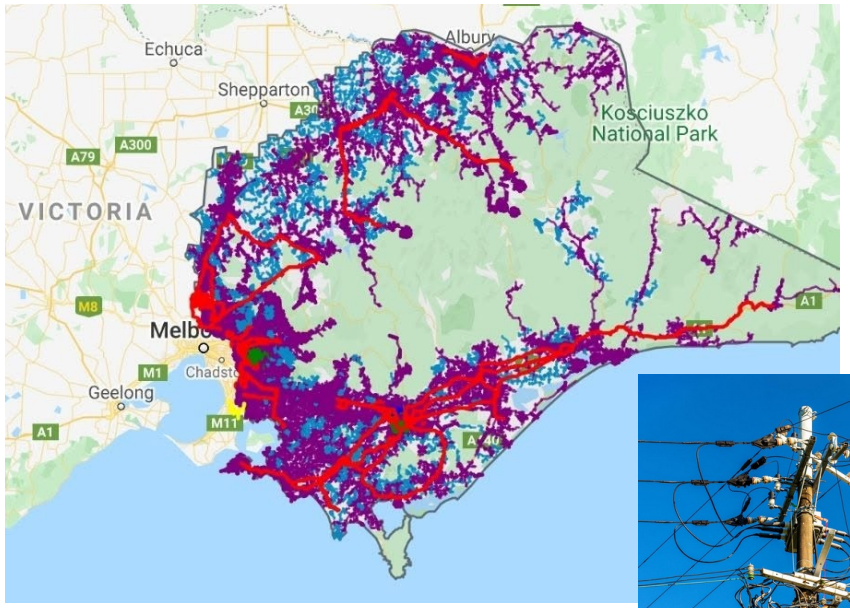
Transmission

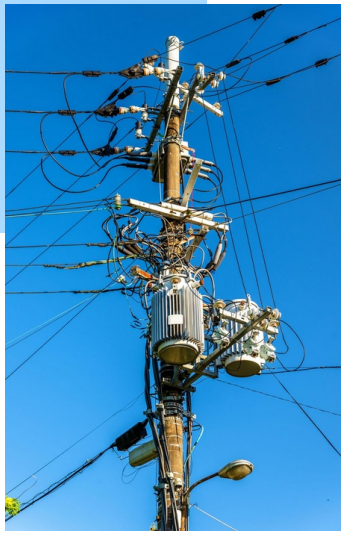




https://www.aemc.gov.au/energy-system/electricity/electricity-system/NEM

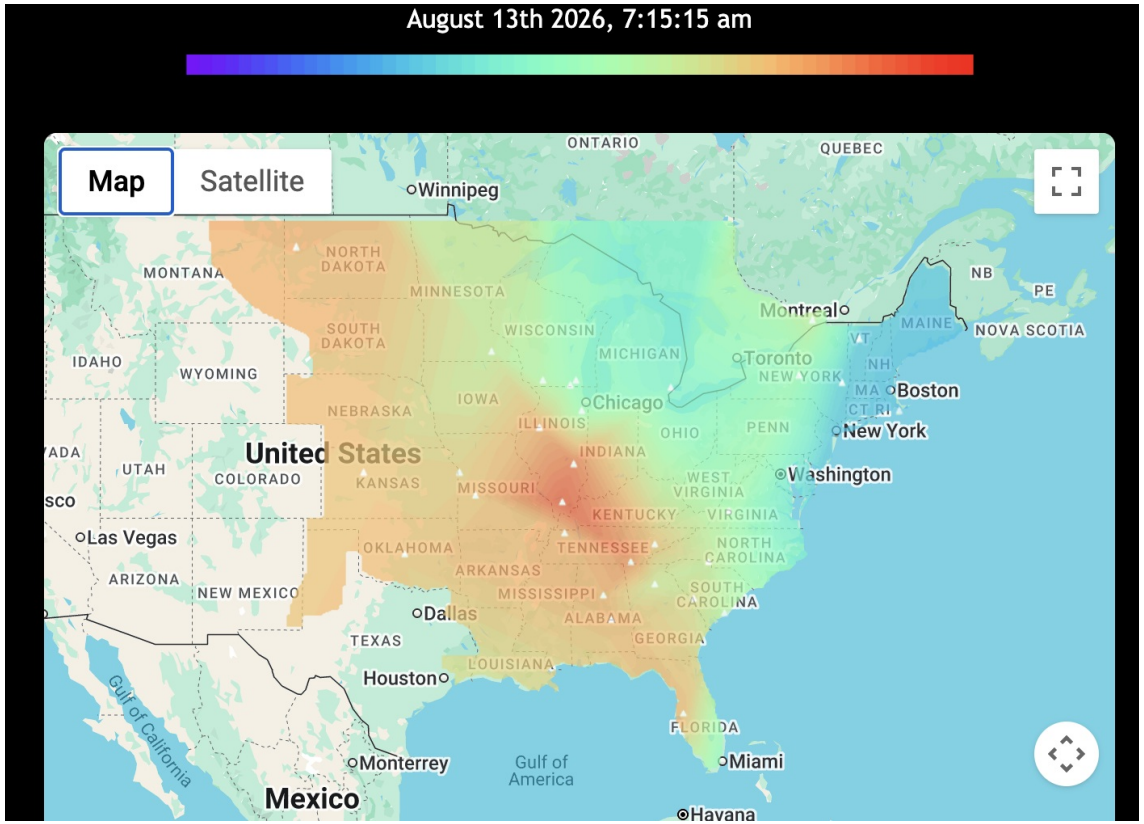
Distribution





https://dapr.ausnetservices.com.au/

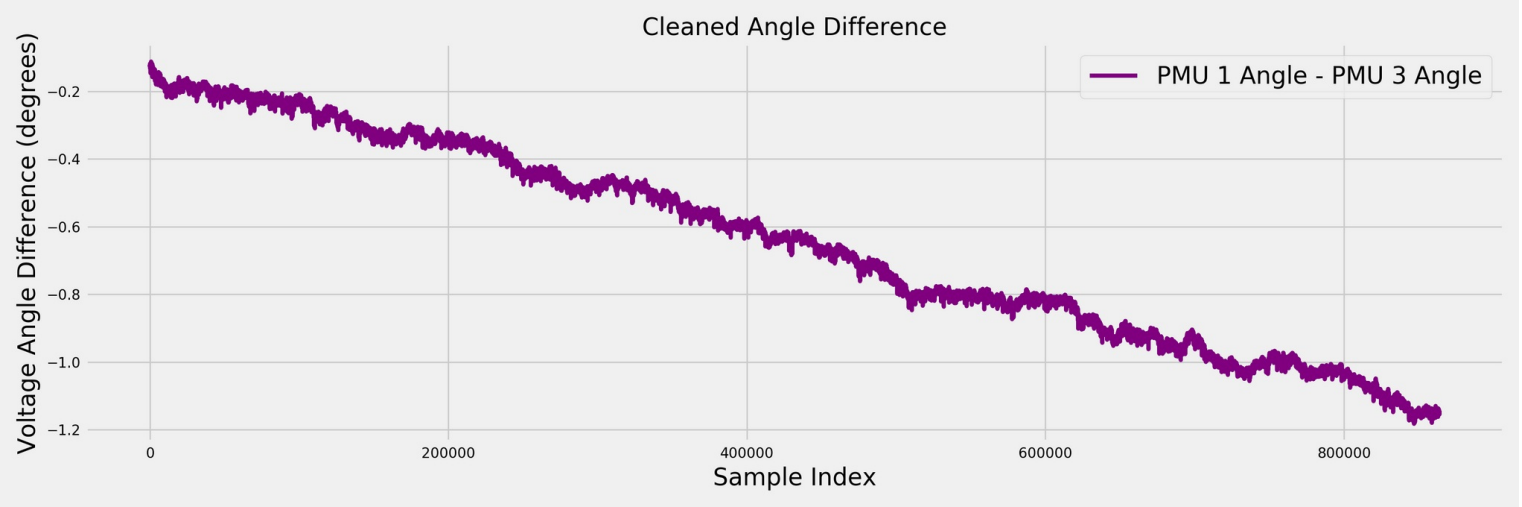
Transmission Network Angle Separation



https://fnetpublic.utk.edu/anglecontour.html

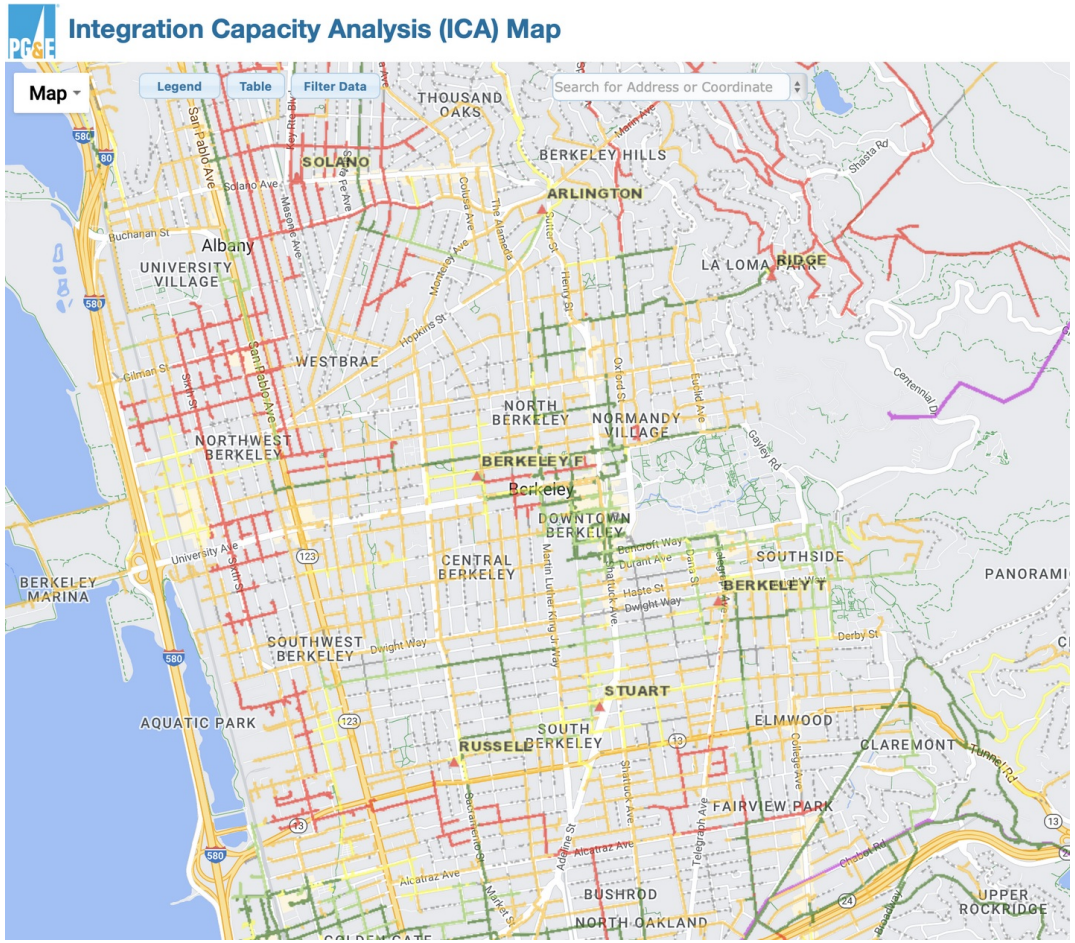
Fnet does not publish the legend values, but we can estimate $\sim 100°$ separation between the red and purple.

Distribution Network Angle Separation



Mohini Bariya and Sascha von Meier https://blog.ni4ai.org/post/2021-04-25-angle-differencing/

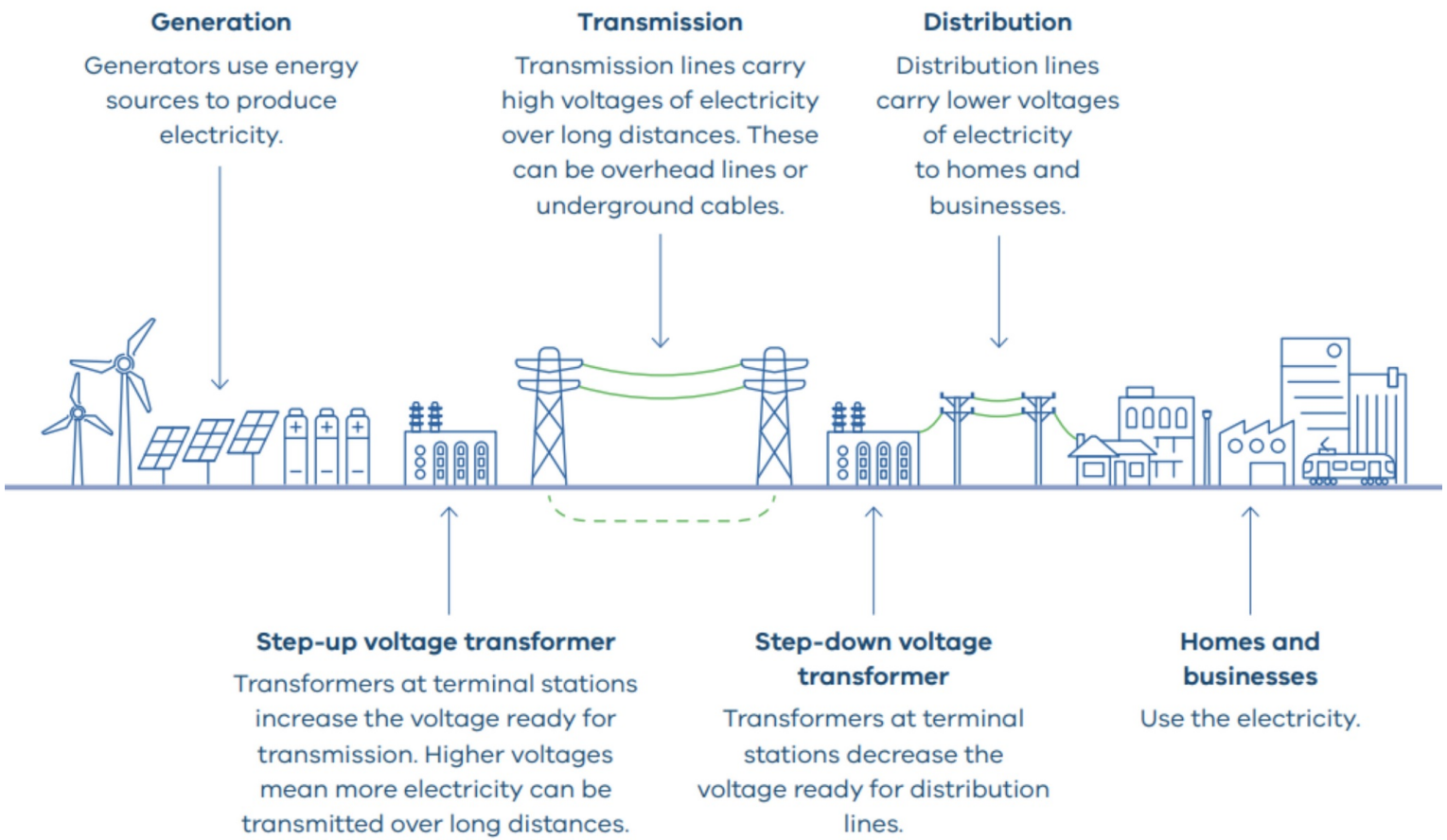
Angle separations on a distribution feeder are usually tenths of a degree, perhaps approaching $\sim 1^{\circ}$ on a long/heavily loaded feeder.



https://www.pge.com/b2b/distribution-resource-planning/integration-capacity-map.shtml

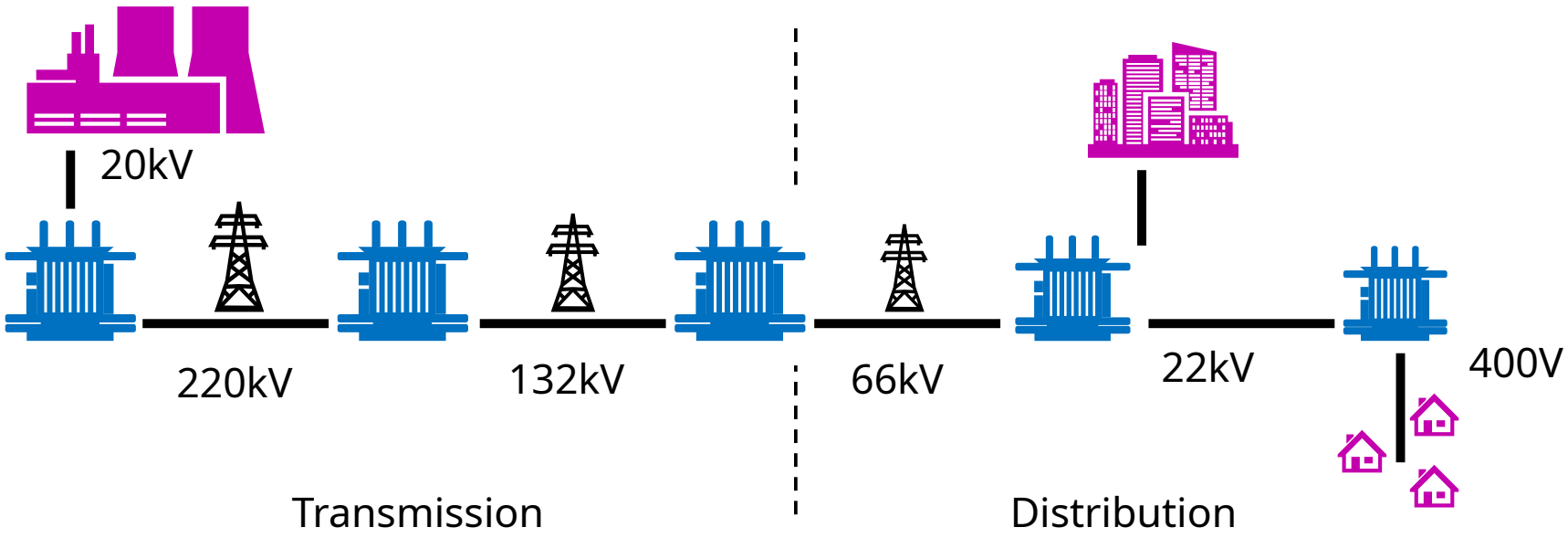
Distribution network capacity determines what you can connect to the grid and where.

Transmission and Distribution



https://www.vicgrid.com.au/transmission-planning/about-energy-infrastructure

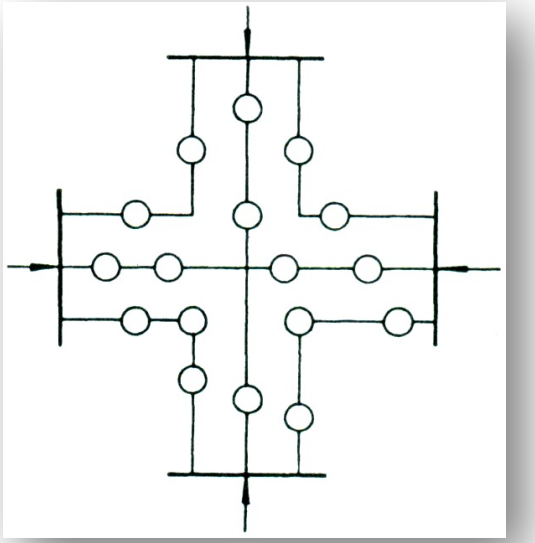
Transmission and Distribution



Voltage levels for Transmission and Distribution are defined by each country or region. There is no universal boundary.

Slide: Prof. Luis (Nando) Ochoa

Transmission and Distribution | Network Configurations

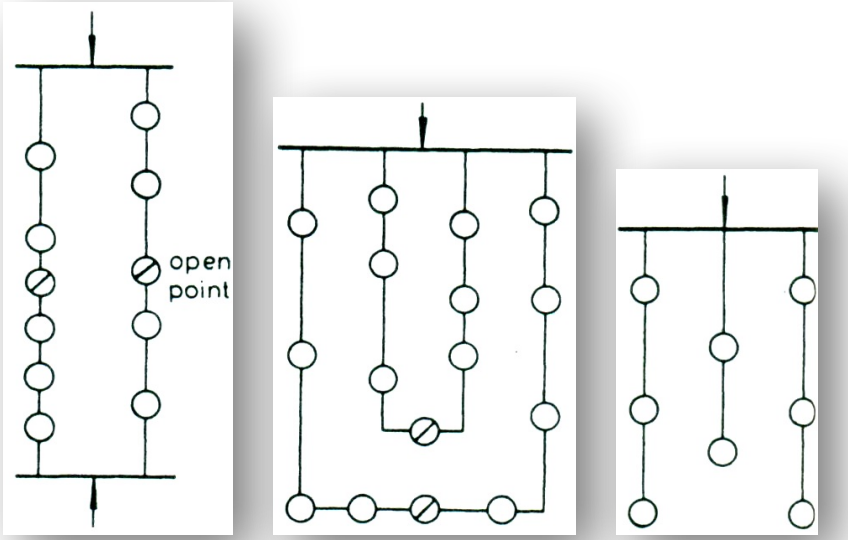


← Transmission

Operated as a meshed/interconnected network

Distribution →

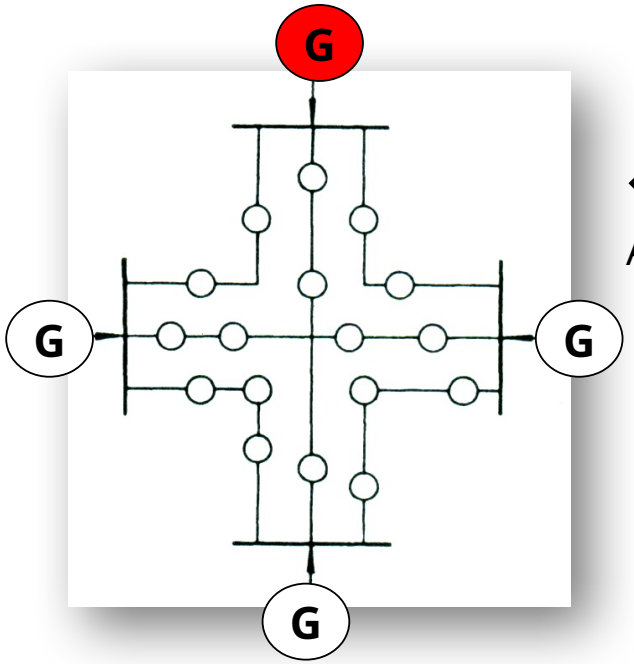
Operated as a radial network



Figures: Electricity distribution network design - Lakervi, E., Holmes, E. J., 1995

Slide: Prof. Luis (Nando) Ochoa

Transmission and Distribution | The Slack Bus

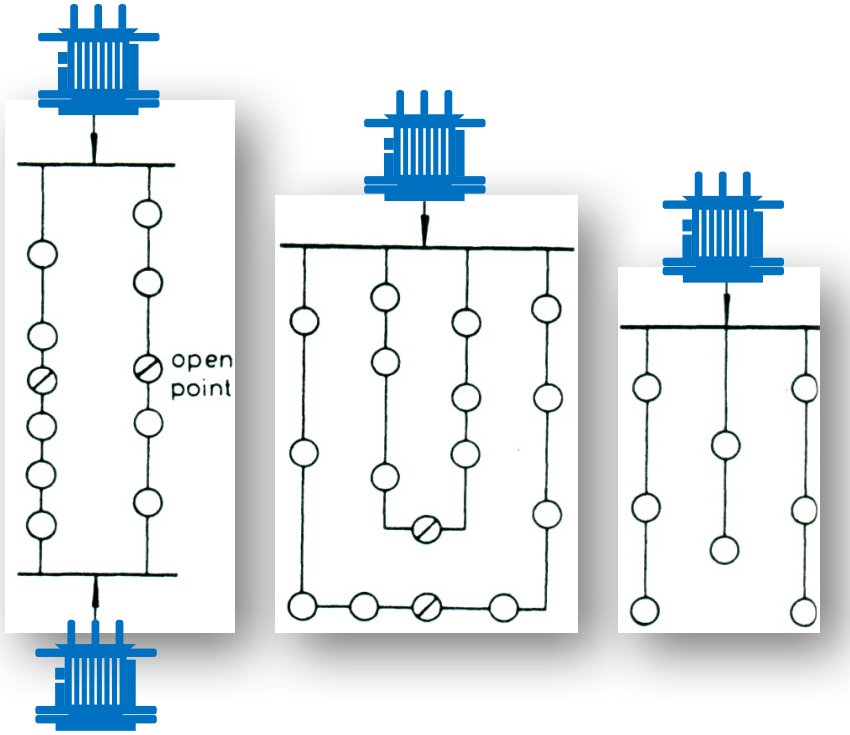


← Transmission

A large generator is typically selected

Distribution →

The substation (or transformer) supplying power is selected.



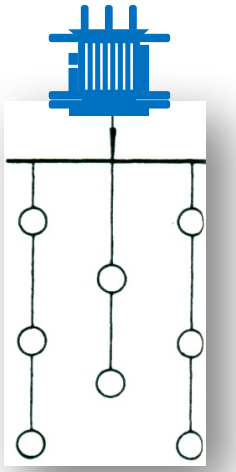
Figures: Electricity distribution network design - Lakervi, E., Holmes, E. J., 1995

Slide: Prof. Luis (Nando) Ochoa

Power Flow in Distribution Networks

Similar to the power flow problem in transmission networks,

- We know the loads
- We want to know $\boldsymbol{V}$ across all buses
  - $\boldsymbol{V}$ tells us power flows through all the lines, losses, etc.



Distribution networks

- Are mainly radially operated
- Have different types of loads than just PQ loads

OpenDSS Documentation

EPRI

1 : Standard constant P+jQ load. (Default) (***dssLoadConstPQ***).
2 : Constant impedance load. (***dssLoadConstZ***).
3 : Const P, Quadratic Q (like a motor). (***dssLoadMotor***).
4 : Nominal Linear P, Quadratic Q (feeder mix). Use this with CVRfactor. (***dssLoadCVR***).

Four of the many OpenDSS load models

Model 1 — Constant P, Q	Standard PQ load	$P = P_{\text{set}}, \quad Q = Q_{\text{set}}$
Model 2 — Constant impedance	Fixed-impedance load	$P, Q \propto V^2$
Model 3 — Constant P, quadratic Q	Motor-like approximation	$P = P_{\text{set}}, \quad Q \propto V^2$
Model 8 — ZIP Load	User-defined model	$P(V) = P_{\text{set}}[a_{Z,P}V^2 + a_{I,P}V + a_{P,P}]$ $\mathrm{Q}(V) = Q_{\text{set}}[a_{Z,Q}V^2 + a_{I,Q}V + a_{P,Q}]$

ZIP Load Model

The ZIP load model allows engineers to combine constant-impedance, constant-current, and constant-power characteristics to approximate load behavior

$$P(V) = P_{\text{set}}\left[a_Z \left(\frac{V}{V_0}\right)^2 + a_I \frac{V}{V_0} + a_P\right]$$

$$a_Z + a_I + a_P = 1$$

Z term	Constant impedance	Power changes with V^2
I term	Constant current	Power changes with V
P term	Constant power	Power does not change with V

The three constants describe how the load's power injections change with voltage magnitude

Equipment/ component	No. tested	V_{cut}	V_0	P_0	Q_0	Z_p	I_p	P_p	Z_q	I_q	P_q
Air compressor 1 Ph	1	100	120	1109.01	487.08	0.71	0.46	-0.17	-1.33	4.04	-1.71
Air compressor 3 Ph	1	174	208	1168.54	844.71	0.24	-0.23	0.99	4.79	-7.61	3.82
Air conditioner	2	100	120	496.33	125.94	1.17	-1.83	1.66	15.68	-27.15	12.47
CFL bulb	2	100	120	25.65	37.52	0.81	-1.03	1.22	0.86	-0.82	0.96
Coffeemaker	1	100	120	1413.04	13.32	0.13	1.62	-0.75	3.89	-6	3.11
Copier	1	100	120	944.23	84.57	0.87	-0.21	0.34	2.14	-3.67	2.53
Electronic ballast	3	100	120	59.02	5.06	0.22	-0.5	1.28	9.64	-21.59	12.95
Elevator	3	174	208	1381.17	1008.3	0.4	-0.72	1.32	3.76	-5.74	2.98
Fan	2	100	120	163.25	83.28	-0.47	1.71	-0.24	2.34	-3.12	1.78
Game consol	3	100	120	60.65	67.61	-0.63	1.23	0.4	0.76	-0.93	1.17
Halogen	3	100	120	97.36	0.84	0.46	0.64	-0.1	4.26	-6.62	3.36
High pressure sodium HID	4	100	120	276.09	52.65	0.09	0.7	0.21	16.6	-28.77	13.17
Incandescent light	2	100	120	87.16	0.85	0.47	0.63	-0.1	0.55	0.38	0.07
Induction light	1	100	120	44.5	4.8	2.96	-6.04	4.08	1.48	-1.29	0.81
Laptop charger	1	100	120	35.94	71.64	-0.28	0.5	0.78	-0.37	1.24	0.13
LCD Television	1	100	120	208.03	-20.58	0.11	-0.17	1.06	1.58	-1.72	1.14
LED light	1	100	120	3.38	5.85	0.58	1.13	-0.71	1.78	-0.8	0.02
Magnetic ballast	1	100	120	81.23	8.2	-1.58	3.79	-1.21	36.18	-67.78	32.6
Mercury vapor HID light	2	100	120	268.27	77.66	0.52	1.02	-0.54	-1.33	2.4	-0.07
Metal halide HID electronic ballast	2	100	120	113.7	26.37	1	-2.02	2.02	8.8	-18.64	10.84
Metal halide HID magnetic ballast	2	100	120	450	102.94	0.86	-0.66	0.8	32.54	-59.83	28.29
Microwave	2	100	120	1365.53	451.02	1.39	-1.96	1.57	50.07	-93.55	44.48
Minibar	1	100	120	90.65	126.94	2.5	-4.1	2.6	2.56	-2.76	1.2
PC (Monitor & CPU)	1	100	120	118.9	172.79	0.2	-0.3	1.1	0	0.6	0.4
Projector	1	100	120	253	44	0.23	-0.52	1.29	0.24	-0.17	0.93
Refrigerator	1	100	120	119.55	52.47	1.17	-1.83	1.66	7.07	-10.94	4.87
Resistive heater	1	100	120	914.78	1.46	0.64	0.59	-0.23	0.13	0.75	0.12
Tungsten light	1	100	120	256.2	21.04	0.43	0.7	-0.13	-0.11	0.66	0.45
Vaccum	1	100	120	855	221	1.18	-0.38	0.2	4.1	-5.87	2.77

Experimental Determination of the ZIP Coefficients..., Bokhari et al
https://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=6648709

Power Electronics Load | Example 1: EV charger

Power electronics loads are constant power, unless they hit their current limit



Normal operation: approximately constant power

$$P = P_{\text{set}}, \qquad Q = 0, \qquad I = \frac{P_{\text{set}}}{V}$$

At $V = 0.9$, constant power needs 11.1% more current

$$I = \frac{I_{nominal}}{0.9} = 1.111 I_{nominal}$$

If that current demand reaches the hardware rating

$$I = I_{max}, \qquad P \approx V I_{max}$$

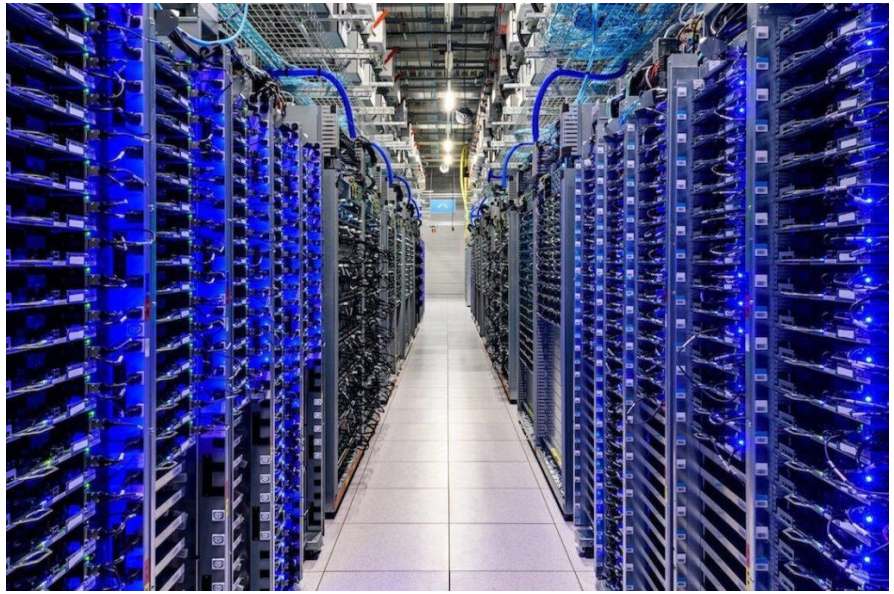
approx. constant-current load, no longer constant power

(P is now proportional to V : $P \propto V$)

At still lower voltage, undervoltage protection may disconnect the load

Power Electronics Load | Example 2: Data Center

Power electronics loads are constant power, unless they hit their current limit



Normal operation: approximately constant power

$$P = P^{\star}, \qquad Q = 0, \qquad I = \frac{P^{\star}}{V}$$

At $V = 0.9$, constant power needs 11.1% more current

$$I = \frac{I_{nominal}}{0.9} = 1.111 I_{nominal}$$

If that current demand reaches the hardware rating

$$I = I_{max}, \qquad P \approx V I_{max}$$

approx. constant-current load, no longer constant power

(P is now proportional to V : $P \propto V$)

At still lower voltage, undervoltage protection may disconnect the load

Two Different AC Power Flow Algorithms

Newton-Raphson

$$-\begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}\begin{bmatrix} \Delta\theta^{(k)} \\ \Delta V^{(k)} \end{bmatrix} = \begin{bmatrix} f_P\left(\theta^{(k)}, V^{(k)}\right) \\ f_Q\left(\theta^{(k)}, V^{(k)}\right) \end{bmatrix}$$

$$\begin{bmatrix} \theta^{(k+1)} \\ V^{(k+1)} \end{bmatrix} = \begin{bmatrix} \theta^{(k)} \\ V^{(k)} \end{bmatrix} + \begin{bmatrix} \Delta\theta^{(k)} \\ \Delta V^{(k)} \end{bmatrix}$$

Fixed-Point Current-Injection

$$\boldsymbol{I}_P^{(k)} = \phi_P\left(\boldsymbol{V}_P^{(k)}\right)$$

$$\boldsymbol{Y}_{PP}\left(\boldsymbol{V}_P^{(k+1)} - 1\right) = \boldsymbol{I}_P^{(k)}$$

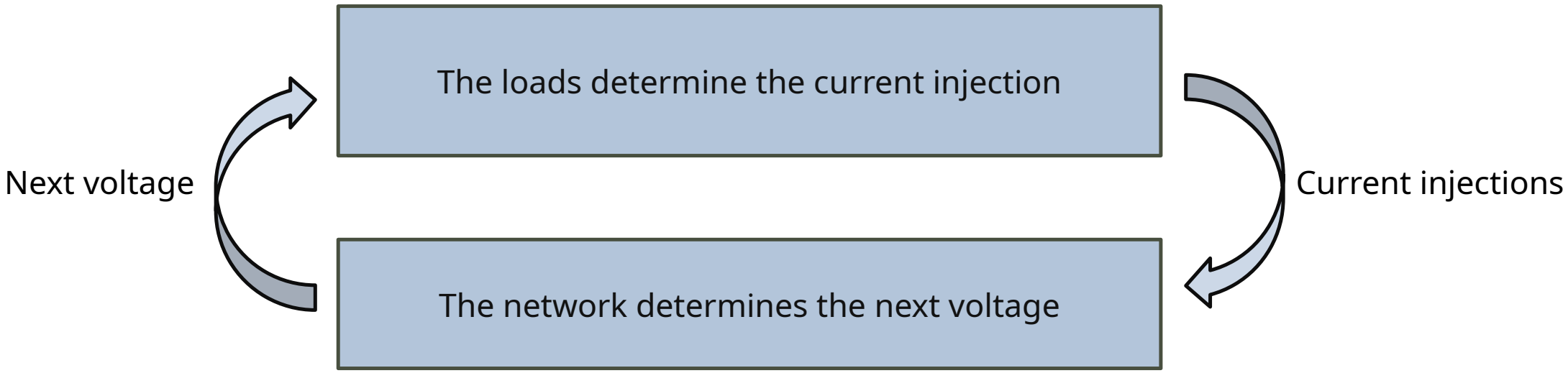
These are the two relevant AC Power Flow algorithms for distribution networks today.

Fixed-Point Current-Injection is a natural fit for Power Flow solves with arbitrary $\boldsymbol{I}_P\left(\boldsymbol{V}_P^{(k)}\right)$.

Newton-Raphson methods can also be used, but determining the Jacobian takes effort.

An intuitive approach based on load modeling

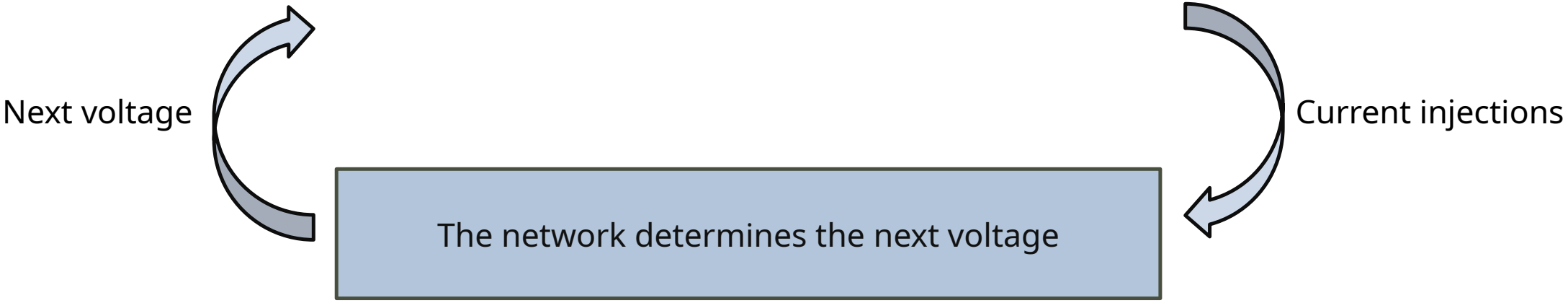
The load currents, at the correct voltage, flow through the network to produce that voltage.



When the voltages stop changing, the criterion above is met.

This is fundamentally different than Newton's Method! Can it work?

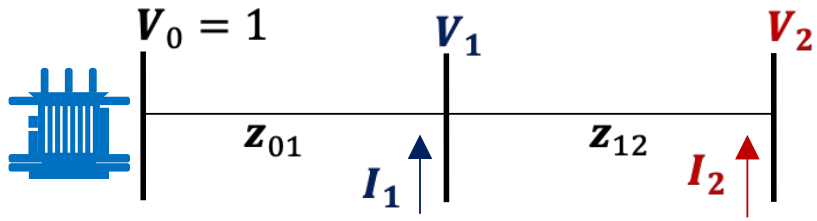
How do you determine the voltages from the current injections?



Class exercise:

Determine the voltages V_1 and V_2 as functions of I_1 and I_2

Hint: What is the current flowing on each branch?



Backward/Forward Sweep

The load characteristic gives the current as a function of voltage:

$$\boxed{\boldsymbol{I}_i^{(k)} = \boldsymbol{\phi}_i\left(\boldsymbol{V}_i^{(k)}\right)}$$

For example, a constant power load is: $\boldsymbol{I}_i^{(k)} = \overline{\left(\frac{\boldsymbol{S}_i}{\boldsymbol{V}_i^{(k)}}\right)}$

The "backward/forward sweep" determines the network voltages, given the current injections.

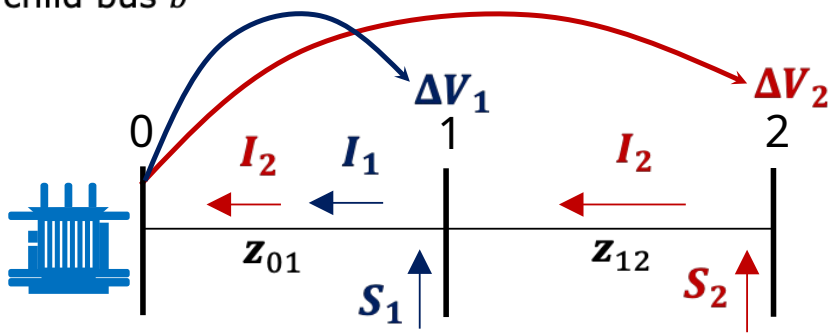
Backward, from all leaves to root/substation:

$$\boldsymbol{I}_{ab}^{(k)} = -\boldsymbol{I}_b\left(\boldsymbol{V}_b^{(k)}\right) + \sum_{\ell \in \text{children}(b)} \boldsymbol{I}_{b\ell}^{(k)}$$

Forward, from root/substation to all leaves:

$$\boldsymbol{V}_h^{(k+1)} = \boldsymbol{V}_i^{(k+1)} - \boldsymbol{z}_{ih}\boldsymbol{I}_{ih}^{(k)}$$

$\boldsymbol{I}_{ab}$: branch current flowing from parent bus a to child bus b



So in this example, $\boldsymbol{I}_{01} = -\boldsymbol{I}_1 - \boldsymbol{I}_2$

Backward/Forward Sweep

Intuition: A power-flow solution is a consistent operating point, the network and the loads agree on currents and voltages

Next voltage

For all loads i: $\boldsymbol{I}_i^{(k)} = \phi_i\left(\boldsymbol{V}_i^{(k)}\right)$

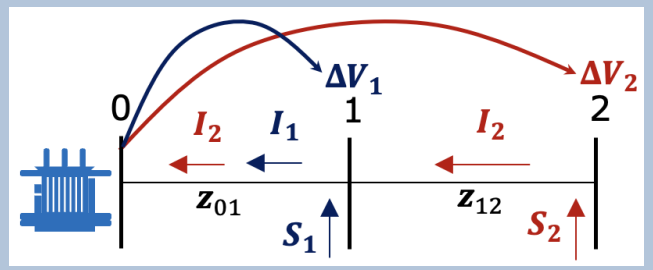
For a constant-power load: $\boldsymbol{I}_i^{(k)} = \overline{\left(\frac{\boldsymbol{S}_i}{\boldsymbol{V}_i^{(k)}}\right)}$

Backward, from all leaves to root/substation:

$$\mathbf{I}_{ab}^{(k)} = -\phi_b\left(\mathrm{V}_b^{(k)}\right) + \Sigma_{\ell \in \text{children}(b)}\mathbf{I}_{b\ell}^{(k)}$$

Forward, from root/substation to all leaves:

$$\boldsymbol{V}_h^{(k+1)} = \boldsymbol{V}_i^{(k+1)} - \boldsymbol{z}_{ih}\boldsymbol{I}_{ih}^{(k)}$$



Current injections

Repeat until the voltage reproduces itself: $\max_i \left|\boldsymbol{V}_i^{(k)} - \boldsymbol{V}_i^{(k-1)}\right| < \epsilon$

Note: **All methods in this lecture use complex-valued currents and voltages**,
unlike Newton-Raphson which uses the real-valued counterparts.

For clarity of explanation, the 3 bus example we examine uses resistive lines and real-power loads.
The methods in this lecture work the same with complex-impedance lines and complex-valued loads.

Backward/Forward Sweep | 3 Bus Example

Backward sweep

$$I_{12}^{(k)} = -I_2^{(k)}$$

$$I_{01}^{(k)} = -I_1^{(k)} + I_{12}^{(k)}$$

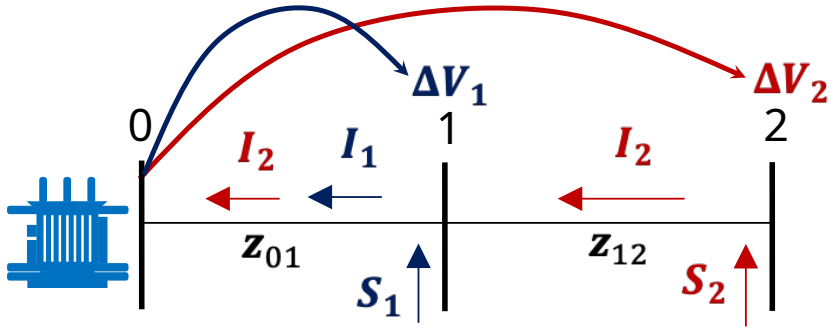
Forward sweep

$$V_1^{(k+1)} = V_0 - z_{01} I_{01}^{(k)}$$

$$V_2^{(k+1)} = V_1^{(k+1)} - z_{12} I_{12}^{(k)}$$

$V^{(0)} = 1, z_{01} = z_{12} = 1,$ $\quad Y_{PP} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$S_1 = S_2 = -0.02$ $\quad I_i^{(k)} = \left(\overline{\frac{-0.02}{V_i^{(k)}}} \right)$



Iterations	Backward	Forward	Check
$I_1^{(0)} = -0.020,$	$I_{12}^{(0)} = 0.020,$	$V_1^{(1)} = 1 - 0.040 = 0.960,$	$\left\lvert V_1^{(1)} - V_1^{(0)} \right\rvert = 0.040$
$I_2^{(0)} = -0.020,$	$I_{01}^{(0)} = 0.040,$	$V_2^{(1)} = 0.960 - 0.020 = 0.940,$	$\left\lvert V_2^{(1)} - V_2^{(0)} \right\rvert = 0.060$
$I_1^{(1)} = \frac{-0.02}{0.960} \approx -0.021,$	$I_{12}^{(1)} = 0.021,$	$V_1^{(2)} = 1 - 0.042 = 0.958,$	$\left\lvert V_1^{(2)} - V_1^{(1)} \right\rvert = 0.002$
$I_2^{(1)} = \frac{-0.02}{0.940} \approx -0.021,$	$I_{01}^{(1)} = 0.042,$	$V_2^{(2)} = 0.958 - 0.021 = 0.937,$	$\left\lvert V_2^{(2)} - V_2^{(1)} \right\rvert = 0.003$

Backward/Forward Sweep | 3 Bus Example

Backward sweep

$$I_{12}^{(k)} = -I_2^{(k)}$$

$$I_{01}^{(k)} = -I_1^{(k)} + I_{12}^{(k)}$$

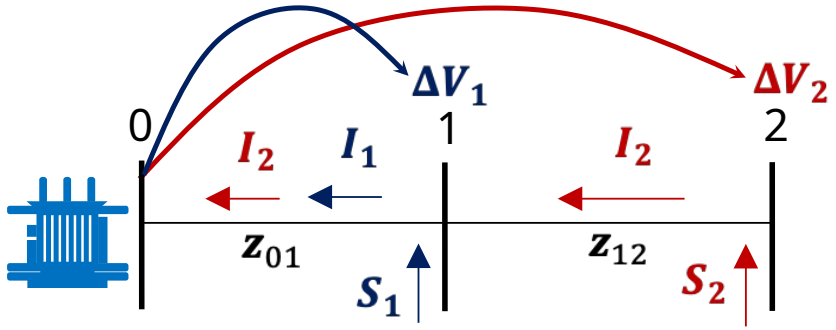
Forward sweep

$$V_1^{(k+1)} = V_0 - z_{01} I_{01}^{(k)}$$

$$V_2^{(k+1)} = V_1^{(k+1)} - z_{12} I_{12}^{(k)}$$

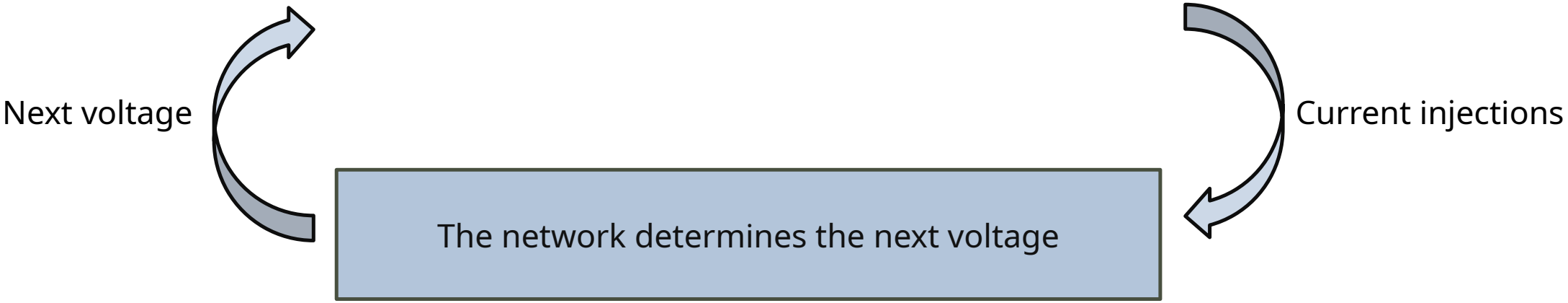
$V^{(0)} = 1, z_{01} = z_{12} = 1,$ $\quad Y_{PP} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$S_1 = S_2 = -0.02$ $\quad I_i^{(k)} = \left(\overline{\frac{-0.02}{V_i^{(k)}}} \right)$



k	$V_1^{(k)}$	$V_2^{(k)}$	$\max_i \left\lvert V_i^{(k)} - V_i^{(k-1)} \right\rvert$
1	0.9600	0.9400	0.0600
2	0.9579	0.9366	0.0034
3	0.9578	0.9364	0.0002
4	0.9578	0.9364	0.0000

How do you determine the voltages from the current injections **if the network is not radial**?



We need to determine the voltages $\boldsymbol{V}_P$ from the current injections $\boldsymbol{I}_P$.

We would like to use $\begin{bmatrix}\boldsymbol{I}_0\\ \boldsymbol{I}_P\end{bmatrix}=\begin{bmatrix}\boldsymbol{Y}_{00} & \boldsymbol{Y}_{0P}\\ \boldsymbol{Y}_{P0} & \boldsymbol{Y}_{PP}\end{bmatrix}\begin{bmatrix}\boldsymbol{V}_0\\ \boldsymbol{V}_P\end{bmatrix}$ but $\boldsymbol{Y}=\begin{bmatrix}\boldsymbol{Y}_{00} & \boldsymbol{Y}_{0P}\\ \boldsymbol{Y}_{P0} & \boldsymbol{Y}_{PP}\end{bmatrix}$ is not invertible.

Fortunately, $\boldsymbol{Y}_{PP}$ is invertible.

Deriving the non-slack admittance matrix equation

(no shunt admittance case)

Bus 0 is the slack bus; subscript P denotes all non-slack buses:

$$\begin{bmatrix} \boldsymbol{I}_0 \\ \boldsymbol{I}_P \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{Y}_{00} & \boldsymbol{Y}_{0P} \\ \boldsymbol{Y}_{P0} & \boldsymbol{Y}_{PP} \end{bmatrix}}_{\boldsymbol{Y}} \begin{bmatrix} \boldsymbol{V}_0 \\ \boldsymbol{V}_P \end{bmatrix}$$

the vector of non-slack-bus current injections

the vector of non-slack-bus voltages

Since $\boldsymbol{Y}1 = 0$, subtracting 1 from every bus voltage does not change the current injections:

$$\begin{bmatrix} \boldsymbol{I}_0 \\ \boldsymbol{I}_P \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y}_{00} & \boldsymbol{Y}_{0P} \\ \boldsymbol{Y}_{P0} & \boldsymbol{Y}_{PP} \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{V}_P - 1 \end{bmatrix}$$

The second row of this equation is:

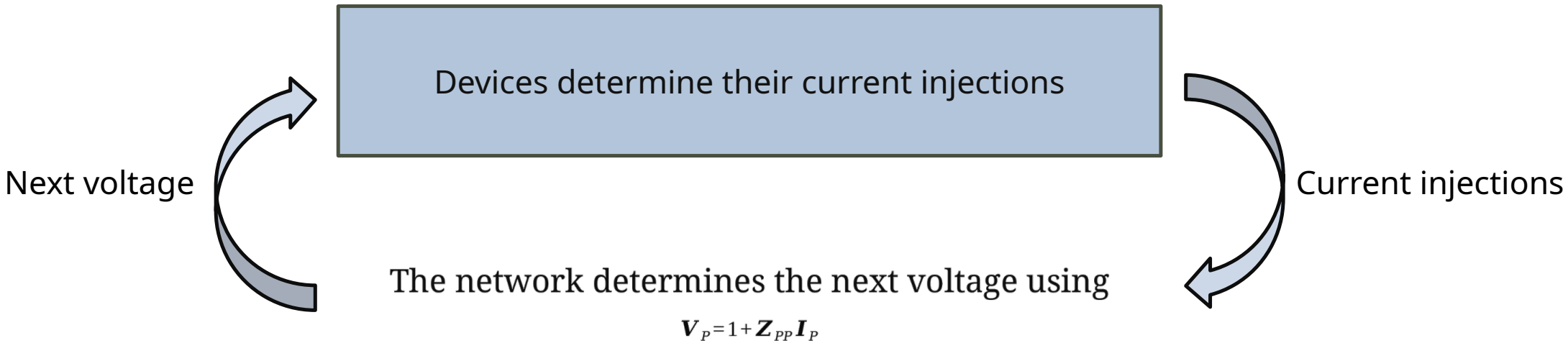
$$\boldsymbol{I}_P = \boldsymbol{Y}_{PP}(\boldsymbol{V}_P - 1)$$

$\boldsymbol{Y}_{PP}$ is full rank and can be inverted, $\boldsymbol{Y}_{PP}^{-1} = \boldsymbol{Z}_{PP}$, thus we have:

$$\boldsymbol{V}_P = 1 + \boldsymbol{Z}_{PP}\boldsymbol{I}_P$$

An intuitive approach to Power Flow

The load currents, at the correct voltage, flow through the network to produce that voltage.



When the voltages stop changing, the criterion above is met.

This is fundamentally different than Newton's Method! Can it work?

History: Z-Bus Power Flow

The Fixed-Point Current-Injection Method is closely related to the Z-bus Method, which inverts $\boldsymbol{Y}_{PP}$ to get

$$\boldsymbol{V}_P^{(k+1)} = 1 + \boldsymbol{Z}_{PP}\boldsymbol{I}_P\left(\boldsymbol{V}_P^{(k)}\right)$$

1963:

Power Flow Solution by Impedance Matrix Iterative Method

H. E. BROWN, SENIOR MEMBER IEEE
G. K. CARTER, FELLOW IEEE
H. H. HAPP, SENIOR MEMBER IEEE
C. E. PERSON, MEMBER IEEE

https://ieeexplore.ieee.org/document/4072757



https://www.ibm.com/history/vanguard

"Our program required 180 seconds [to run the 124-bus AC power flow on a vacuum-tube IBM 704]. For each system being studied the matrix is formed only once and is kept in core throughout the entire study."

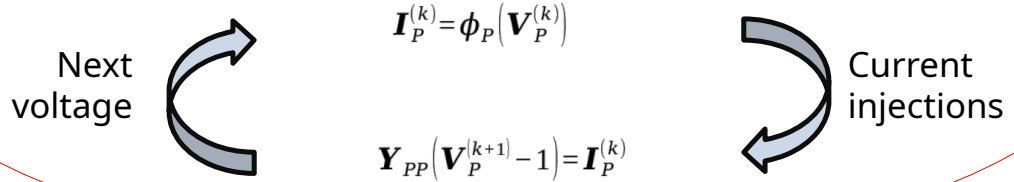
Same Fixed-Point Iteration, Two Implementations

Bus 0 is the slack bus; subscript P denotes all non-slack buses:

$$\begin{bmatrix} \boldsymbol{I}_0 \\ \boldsymbol{I}_P \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y}_{00} & \boldsymbol{Y}_{0P} \\ \boldsymbol{Y}_{P0} & \boldsymbol{Y}_{PP} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_0 \\ \boldsymbol{V}_P \end{bmatrix}$$

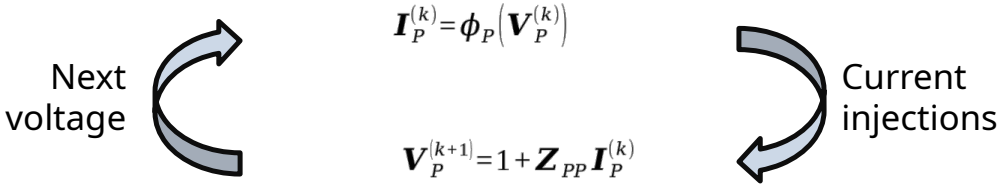
$$\boldsymbol{Z}_{PP} = \boldsymbol{Y}_{PP}^{-1}$$

The Fixed-Point Current-Injection Method



factorize $\boldsymbol{Y}_{PP}$ once, then solve with those factors each iteration

The Z-Bus Method



Requires dense $\boldsymbol{Z}_{PP}$ matrix multiplications

Which is better?

Fixed-Point Power Flow History

1967 intuition: do not form $\boldsymbol{Z} = \boldsymbol{Y}^{-1}$; exploit sparse $\boldsymbol{Y}$ and reuse its factorization

Direct Solutions of Sparse Network Equations by Optimally Ordered Triangular Factorization

WILLIAM F. TINNEY, SENIOR MEMBER, IEEE, AND JOHN W. WALKER, MEMBER, IEEE

Later applied to the Z-bus method for distribution network power flow:

IEEE Transactions on Power Apparatus and Systems, Vol. PAS-99, No. 4 July/Aug 1980

CALCULATION OF ENERGY LOSSES IN A DISTRIBUTION SYSTEM

D.I.H. Sun Non-Member
S. Abe Student Member, IEEE
R.R. Shoults Member, IEEE
M.S. Chen Fellow, IEEE

Energy Systems Research Center
The University of Texas at Arlington
Arlington, Texas 76019

P. Eichenberger Member, IEEE
D. Farris Member, IEEE

Public Service Company of New Mexico
Albuquerque, New Mexico 87103

IEEE Transactions on Power Delivery, Vol. 6, No. 3, July 1991

DISTRIBUTION SYSTEM POWER FLOW ANALYSIS - A Rigid Approach

Tsai-Hsiang Chen Member, IEEE
Mo-Shing Chen Fellow, IEEE
Kab-Ju Hwang

Energy Systems Research Center
The University of Texas at Arlington
Arlington, Texas 76019

Paul Kotas
Elie A. Chebli Member, IEEE

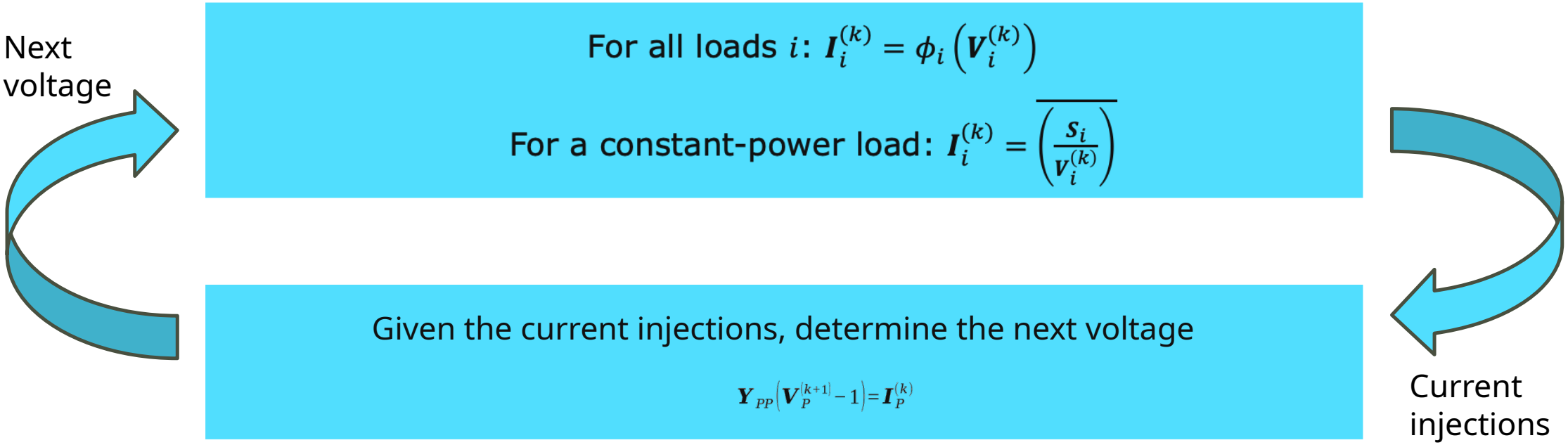
Consolidated Edison Company
of New York, Inc.
4 Irving Place, New York, NY 10003

The same sparse-Y insight is used by the open source, OpenDSS software for Distribution Network power flow



The Fixed-Point Current-Injection Method

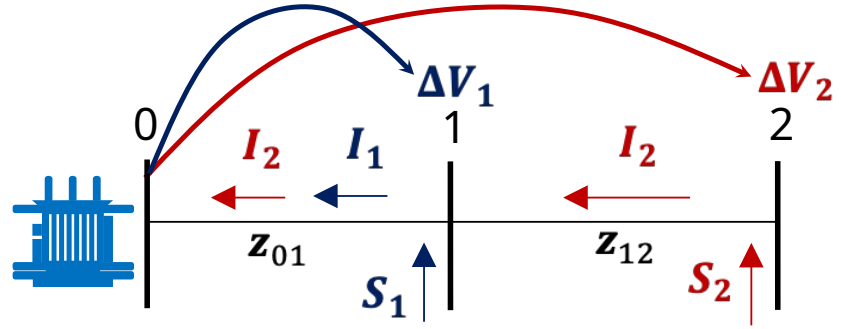
Intuition: A power-flow solution is a consistent operating point, the network and the loads agree on currents and voltages

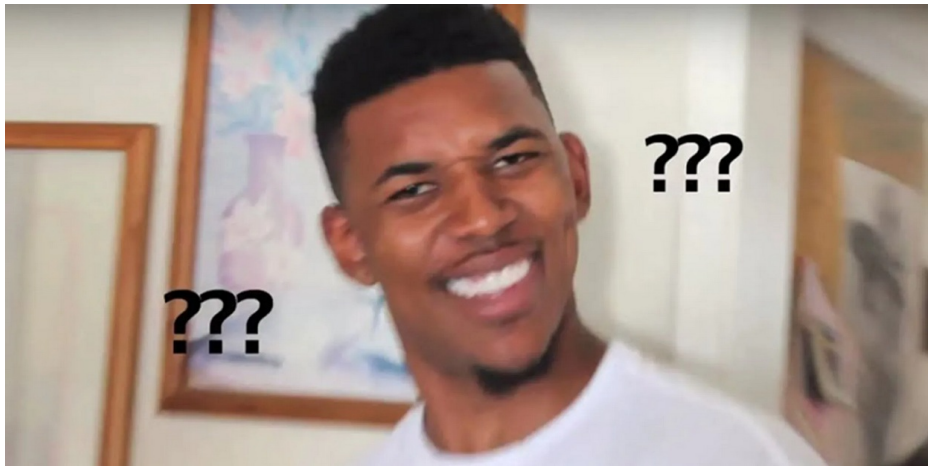


Repeat until the voltage reproduces itself: $\max_i \left|\boldsymbol{V}_i^{(k)} - \boldsymbol{V}_i^{(k-1)}\right| < \epsilon$

It turns out, Backward/Forward Sweep provides intuition for what sparse matrix solvers do.

Sparse solve of $\boldsymbol{Y}_{PP}\left(\boldsymbol{V}_P^{(k+1)}-1\right)=\boldsymbol{I}_P^{(k)}$





Admittance Matrix Decomposition

(still no shunt admittances)

Where does the formula for the admittance matrix $\boldsymbol{Y}$ come from?

A: Bus Incidence Matrix

rows = branches; columns = buses

for row/branch ℓ from parent h to child i:

$A_{\ell h} = +1, \qquad A_{\ell i} = -1, \qquad A_{\ell k} = 0 \ (k \neq h, i)$

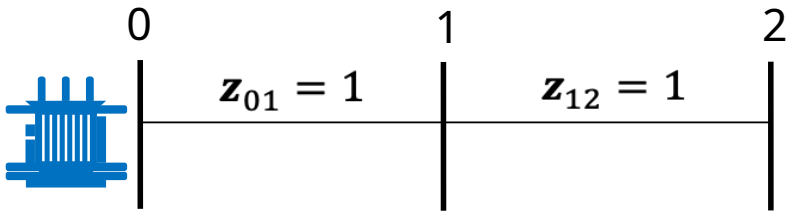
$\boldsymbol{Y}_{\mathrm{br}}$: Branch Admittance Matrix

$$\boldsymbol{Y}_{\mathrm{br}} = \mathrm{diag}(\boldsymbol{y_1}, \ldots, \boldsymbol{y_m}), \qquad \boldsymbol{y}_\ell = \boldsymbol{z}_\ell^{-1}$$

$\boldsymbol{V}_{\mathrm{br}}$: the vector of voltage differences across each line
$\boldsymbol{I}_{\mathrm{br}}$: the vector of currents through each line

1. $\boldsymbol{V}_{\mathrm{br}} = A\boldsymbol{V}$ (Kirchoff's Voltage Law)
2. $\boldsymbol{I}_{\mathrm{br}} = \boldsymbol{Y}_{\mathrm{br}}\boldsymbol{V}_{\mathrm{br}}$ (Ohm's Law)
3. $\boldsymbol{I} = A^T\boldsymbol{I}_{\mathrm{br}}$ (Kirchoff's Current Law)

Admittance Matrix Decomposition | 3 Bus Example



A: Bus incidence matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Y_{br} : Branch admittance matrix

$$Y_{\text{br}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_{\text{br}} = \begin{bmatrix} V_0 - V_1 \\ V_1 - V_2 \end{bmatrix}, \quad I_{\text{br}} = \begin{bmatrix} I_{01} \\ I_{12} \end{bmatrix}$$

1. $V_{\text{br}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} V$

2. $I_{\text{br}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} V_{\text{br}}$

3. $I = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} I_{\text{br}}$

Admittance Matrix Decomposition

(still no shunt admittances)

Where does the formula for the admittance matrix $\boldsymbol{Y}$ come from?

A: Bus Incidence Matrix

rows = branches; columns = buses

for row/branch ℓ from parent h to child i:

$A_{\ell h} = +1, \qquad A_{\ell i} = -1, \qquad A_{\ell k} = 0 \ (k \neq h, i)$

$\boldsymbol{Y}_{\mathrm{br}}$: Branch Admittance Matrix

$$\boldsymbol{Y}_{\mathrm{br}} = \mathrm{diag}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m), \qquad \boldsymbol{y}_\ell = \boldsymbol{z}_\ell^{-1}$$

$\boldsymbol{V}_{\mathrm{br}}$: the vector of voltage differences across each line
$\boldsymbol{I}_{\mathrm{br}}$: the vector of currents through each line

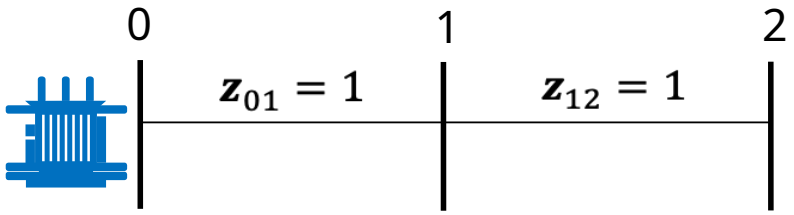
1. $\boldsymbol{V}_{\mathrm{br}} = A\boldsymbol{V}$ (KVL)

2. $\boldsymbol{I}_{\mathrm{br}} = \boldsymbol{Y}_{\mathrm{br}}\boldsymbol{V}_{\mathrm{br}}$ (Ohm's Law)

3. $\boldsymbol{I} = A^T\boldsymbol{I}_{\mathrm{br}}$ (KCL)

Combining 1, 2, and 3: $\boldsymbol{I} = \underbrace{A^T\boldsymbol{Y}_{\mathrm{br}}A}_{\boldsymbol{Y}}\boldsymbol{V}$

Admittance Matrix Decomposition | 3 Bus Example



$$\boldsymbol{Y} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

A: Bus incidence matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$\boldsymbol{Y}_{\text{br}}$: Branch admittance matrix

$$\boldsymbol{Y}_{\text{br}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T \boldsymbol{Y}_{\text{br}} A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Admittance Matrix Decomposition

(still no shunt admittances)

Where does the formula for the admittance matrix $\boldsymbol{Y}$ come from?

A: Bus Incidence Matrix

rows = branches; columns = buses

for row/branch ℓ from parent h to child i:

$A_{\ell h} = +1, \qquad A_{\ell i} = -1, \qquad A_{\ell k} = 0 \ (k \neq h, i)$

$\boldsymbol{Y}_{\mathrm{br}}$: Branch Admittance Matrix

$\boldsymbol{Y}_{\mathrm{br}} = \mathrm{diag}(\boldsymbol{y_1}, \ldots, \boldsymbol{y_m}), \qquad \boldsymbol{y}_\ell = \boldsymbol{z}_\ell^{-1}$

The admittance matrix is $\boldsymbol{Y} = A^T \boldsymbol{Y}_{br} A$

The map from $\boldsymbol{V}$ (the bus voltages) to $\boldsymbol{V}_{\mathrm{br}}$: A

The map from $\boldsymbol{V}_{\mathrm{br}}$ to $\boldsymbol{I}_{\mathrm{br}}$: $\boldsymbol{Y}_{br}$

The map from $\boldsymbol{I}_{\mathrm{br}}$ to $\boldsymbol{I}$ (the bus current injections): A^T

} $\boldsymbol{Y}$: the map from $\boldsymbol{V}$ to $\boldsymbol{I}$

Reduced Admittance Matrix Decomposition

(still no shunt admittances)

A: Bus Incidence Matrix

rows = branches; columns = buses

for row/branch ℓ from parent h to child i:

$A_{\ell h} = +1, \qquad A_{\ell i} = -1, \qquad A_{\ell k} = 0 \ (k \neq h, i)$

A_P: A with the slack-bus column removed

$\boldsymbol{Y}_{\mathrm{br}}$: Branch Admittance Matrix

$$\boldsymbol{Y}_{\mathrm{br}} = \mathrm{diag}(\boldsymbol{y_1}, \ldots, \boldsymbol{y_m}), \qquad \boldsymbol{y}_\ell = \boldsymbol{z}_\ell^{-1}$$

The reduced admittance matrix is $\boldsymbol{Y}_{PP} = A_P^T \boldsymbol{Y}_{br} A_P$

Derivation of $\boldsymbol{Y}_{PP} = A_P^T \boldsymbol{Y}_{br} A_P$

A_P is A with the slack-bus column a_0 removed: $A = [a_0 \; A_P]$

$$\boldsymbol{Y} = \begin{bmatrix} a_0^\top \\ A_P^\top \end{bmatrix} \boldsymbol{Y}_{br} \begin{bmatrix} a_0 & A_P \end{bmatrix}$$

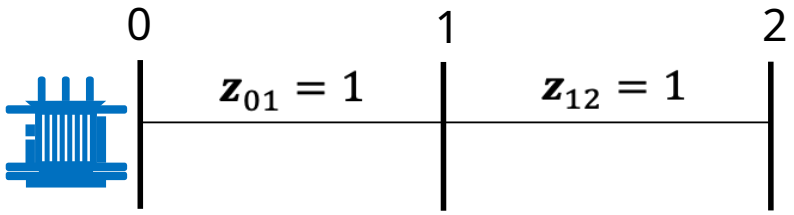
Multiplying out:

$$\boldsymbol{Y} = \begin{bmatrix} a_0^\top \boldsymbol{Y}_{br} a_0 & a_0^\top \boldsymbol{Y}_{br} A_P \\ A_P^\top \boldsymbol{Y}_{br} a_0 & A_P^\top \boldsymbol{Y}_{br} A_P \end{bmatrix}$$

Thus, from $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_{00} & \boldsymbol{Y}_{0P} \\ \boldsymbol{Y}_{P0} & \boldsymbol{Y}_{PP} \end{bmatrix}$,

$$\boldsymbol{Y}_{PP} = A_P^\top \boldsymbol{Y}_{br} A_P$$

Admittance Matrix Decomposition | 3 Bus Example



$$\boldsymbol{Y} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Longrightarrow \boldsymbol{Y}_{PP} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

A: Bus incidence matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

remove slack column 0:

$$A_P = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

$\boldsymbol{Y}_{\mathrm{br}}$: Branch admittance matrix

$$\boldsymbol{Y}_{\mathrm{br}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Demonstration of recovering $\boldsymbol{Y}_{PP}$:

$$A_P^T \boldsymbol{Y}_{br} A_P = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

The Fixed-Point Current-Injection Method

The bus voltage update

Given the current injections, determine the next voltage

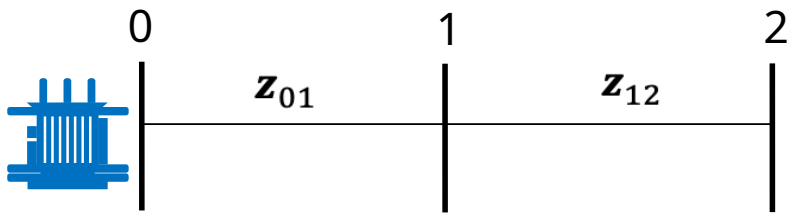
$$\boldsymbol{Y}_{PP}\left(\boldsymbol{V}_P^{(k+1)}-1\right)=\boldsymbol{I}_P^{(k)}$$

is equivalent to

Given the current injections, determine the next voltage

$$A_P^T\boldsymbol{Y}_{br}A_P\left(\boldsymbol{V}_P^{(k+1)}-1\right)=\boldsymbol{I}_P^{(k)}$$

$$\boldsymbol{Y}_{PP} = A_P^T \boldsymbol{Y}_{br} A_P$$



For radial networks, A_P is an invertible matrix.

Proof:

There are $n-1$ branches so A_P is square.

Ordering buses from the root outward makes A_P lower-triangular: $A_P = \begin{bmatrix} -1 & 0 & 0 & \cdots \\ * & -1 & 0 & \cdots \\ * & * & -1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

$det(A_P) = (-1)^{n-1} \neq 0$ so A_P is invertible.

For radial networks,

$$\boldsymbol{Y}_{PP}^{-\mathbf{1}} = A_P^{-1} \boldsymbol{Y}_{br}^{-1} (A_P^T)^{-1}$$

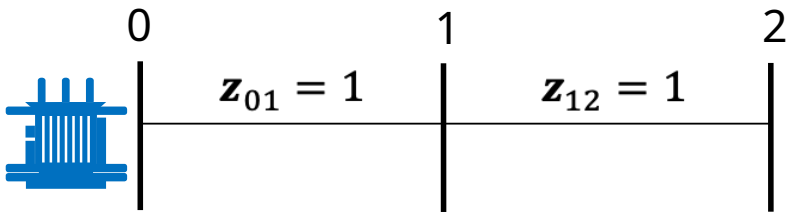
$$= C^T \boldsymbol{Z}_{br} C \qquad \text{where } C^T = A_P^{-1},\ \boldsymbol{Z}_{br} = \boldsymbol{Y}_{br}^{-1}, \text{ and } C = (A_P^T)^{-1}$$

Thus, $A_P^T \boldsymbol{Y}_{br} A_P \left(\boldsymbol{V}_P^{(k+1)} - 1 \right) = \boldsymbol{I}_P^{(k)}$ is equivalent to $\boldsymbol{V}_P^{(k+1)} - 1 = C^T \boldsymbol{Z}_{br} C \, \boldsymbol{I}_P^{(k)}$

Thus, we use

Given the current injections, determine the next voltage

$$\boldsymbol{V}_P^{(k+1)} = 1 + C^T \boldsymbol{Z}_{br} C \, \boldsymbol{I}_P^{(k)}$$



For radial networks,

$C=(A_P^T)^{-1}$: Accumulation matrix

$$C = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

$\boldsymbol{Z}_{\mathrm{br}} = \boldsymbol{Y}_{\mathrm{br}}^{-1}$: Branch impedance matrix

$$\boldsymbol{Z}_{\mathrm{br}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Backward current sweep:

$$\boldsymbol{I}_{br} = C\,\boldsymbol{I}_P$$

$$\begin{bmatrix} \boldsymbol{I}_{01} \\ \boldsymbol{I}_{12} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \end{bmatrix} = \begin{bmatrix} -(\boldsymbol{I}_1 + \boldsymbol{I}_2) \\ -\boldsymbol{I}_2 \end{bmatrix}$$

Forward voltage sweep:

$$\boldsymbol{V}_P = 1 + C^T \boldsymbol{Z}_{br} \boldsymbol{I}_{br}$$

$$\begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_{01} \\ \boldsymbol{I}_{12} \end{bmatrix} = \begin{bmatrix} \boldsymbol{V}_0 - \boldsymbol{I}_{01} \\ \boldsymbol{V}_0 - \boldsymbol{I}_{01} - \boldsymbol{I}_{12} \end{bmatrix} = \begin{bmatrix} \boldsymbol{V}_0 - \boldsymbol{I}_{01} \\ \boldsymbol{V}_1 - \boldsymbol{I}_{12} \end{bmatrix}$$

For radial networks,

Given the current injections, determine the next voltage

$$\boldsymbol{V}_P^{[k+1]} = 1 + C^T \boldsymbol{Z}_{br} C \boldsymbol{I}_P^{(k)}$$

Point 1: The (sparse!) matrix-vector multiplications $C^T \boldsymbol{Z}_{br} C \boldsymbol{I}_P^{(k)}$ are easier than $\boldsymbol{Z}_{PP} \boldsymbol{I}_P^{(k)}$

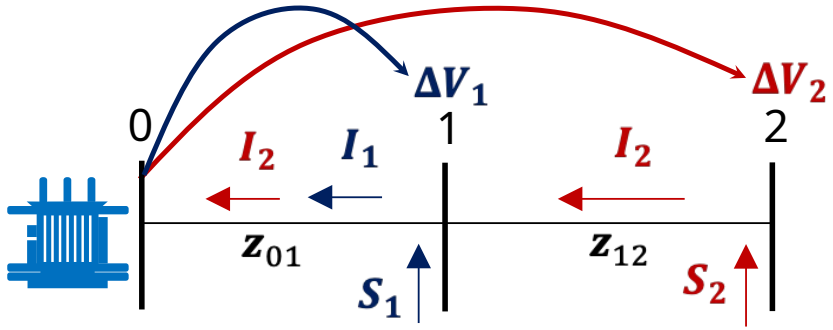
Point 2: solving $\boldsymbol{V}_P^{[k+1]} = 1 + C^T \boldsymbol{Z}_{br} C \boldsymbol{I}_P^{(k)}$ with matrix-vector multiplication *is* the backward-forward sweep algorithm.

Backward sweep

Forward sweep

Meaning it runs all the same operations

$$\boldsymbol{Y}_{PP} = A_P^T \boldsymbol{Y}_{br} A_P$$
$$\boldsymbol{Y}_{PP}^{-1} = C^T \boldsymbol{Z}_{br} C$$



For **radial networks**:

Only work for radial networks

Solving for $\mathbf{V}_P^{(k+1)}$ *from* $\mathbf{I}_P^{(k)}$ *using Backward/Forward Sweep*

equivalent

Solving $\mathbf{Y}_{PP}\left(\mathbf{V}_P^{(k+1)} - 1\right) = \mathbf{I}_P^{(k)}$ *by LDL-factorizing* $\mathbf{Y}_{PP}$

Works for any network

equivalent

solving $\mathbf{V}_P^{(k+1)} = 1 + C^T \mathbf{Z}_{br} C \mathbf{I}_P^{(k)}$ *with matrix-vector multiplication*

Any Network (not just radial)

Given the current injections, determine the next voltage

$$\boldsymbol{Y}_{PP}\left(\boldsymbol{V}_P^{(k+1)} - 1\right) = \boldsymbol{I}_P^{(k)}$$

A_P is not invertible so the equation above is solved for $\boldsymbol{V}_P^{(k+1)}$ after first LDL-decomposing $\boldsymbol{Y}_{PP}$

This involves linear algebra methods that are outside-of-scope for this class.

Important takeaways

For radial networks:

A_P *is invertible, thus we can use* $C^T \mathbf{Z}_{br} C = A_P^{-1} \mathbf{Y}_{br}^{-1} (A_P^T)^{-1}$

Solving $\mathbf{V}_P^{(k+1)} = 1 + C^T \mathbf{Z}_{br} C \mathbf{I}_P^{(k)}$ *with matrix-vector multiplication is efficient.*

The Backward/Forward Sweep algorithm provides physical intuition for $C^T \mathbf{Z}_{br} C \mathbf{I}_P^{(k)}$

For non-radial networks:

A_P *is not invertible*

Solving $\mathbf{Y}_{PP}\left(\mathbf{V}_P^{(k+1)} - 1\right) = \mathbf{I}_P^{(k)}$ *using matrix factorization is still efficient.*

There is no easy physical intuition for what the matrix factorization is doing.

The Fixed-Point Current-Injection Method

Intuition: A power-flow solution is a consistent operating point, the network and the loads agree on currents and voltages

Next voltage

For all loads i: $\boldsymbol{I}_i^{(k)} = \phi_i\left(\boldsymbol{V}_i^{(k)}\right)$

For a constant-power load: $\boldsymbol{I}_i^{(k)} = \overline{\left(\frac{\boldsymbol{S}_i}{\boldsymbol{V}_i^{(k)}}\right)}$

Given the current injections, determine the next voltage

$$\boldsymbol{Y}_{PP}\left(\boldsymbol{V}_P^{(k+1)} - 1\right) = \boldsymbol{I}_P^{(k)}$$

Current injections

Repeat until the voltage reproduces itself: $\max_i \left|\boldsymbol{V}_i^{(k)} - \boldsymbol{V}_i^{(k-1)}\right| < \epsilon$

Relationship Between Methods

Gauss-Seidel

$$V_h^{(k+1)} = 1 + \frac{1}{Y_{hh}}\left(I_h^{(k)} - \sum_{i=1}^{h-1} Y_{hi}\left(V_i^{(k+1)} - 1\right) - \sum_{i=h+1}^{n-1} Y_{hi}\left(V_i^{(k)} - 1\right)\right)$$

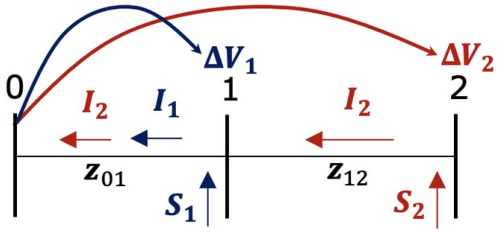


Closely related
GS avoids linear solve and matrix factorization

Backward/Forward Sweep

$$\mathbf{I}_{ab}^{(k)} = -\mathbf{I}_b\left(\mathbf{V}_b^{(k)}\right) + \Sigma_{\ell \in \text{children}(b)} \mathbf{I}_{b\ell}^{(k)}$$

$$V_h^{(k+1)} = V_i^{(k+1)} - z_{ih} I_{ih}^{(k)}$$



Equivalent when network is radial

$$I_P^{(k)} = \phi_P\left(V_P^{(k)}\right)$$

$$Y_{PP}\left(V_P^{(k+1)} - 1\right) = I_P^{(k)}$$

Fixed-Point Current-Injection Method

DSS

Note: **All methods in this lecture use complex-valued currents and voltages**,
unlike Newton-Raphson which uses the real-valued counterparts.

For clarity of explanation, the 3 bus example we examine uses resistive lines and real-power loads.
The methods in this lecture work the same with complex-impedance lines and complex-valued loads.

Why is Fixed-Point Current Injection used for distribution networks?

Load modeling in the Fixed-Point Current-Injection method is straightforward.

Power Flow in Distribution Networks

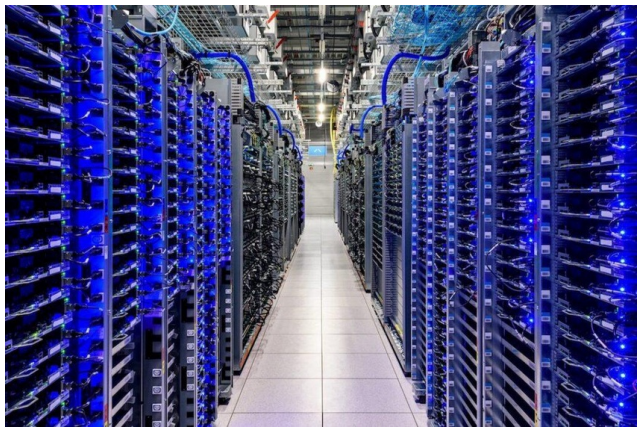
Similar to the power flow problem in transmission networks,

- We know the loads
- We want to know $\boldsymbol{V}$ across all buses
  - $\boldsymbol{V}$ tells us power flows through all the lines, losses, etc.

Distribution networks

- Are mainly radially operated
- Have different types of loads than just PQ loads





Two Different AC Power Flow Algorithms

Newton-Raphson

$$-\begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}\begin{bmatrix} \Delta\theta^{(k)} \\ \Delta V^{(k)} \end{bmatrix} = \begin{bmatrix} f_P\left(\theta^{(k)}, V^{(k)}\right) \\ f_Q\left(\theta^{(k)}, V^{(k)}\right) \end{bmatrix}$$

$$\begin{bmatrix} \theta^{(k+1)} \\ V^{(k+1)} \end{bmatrix} = \begin{bmatrix} \theta^{(k)} \\ V^{(k)} \end{bmatrix} + \begin{bmatrix} \Delta\theta^{(k)} \\ \Delta V^{(k)} \end{bmatrix}$$

(and its variants)

Fixed-Point Current-Injection

$$\boldsymbol{I}_P^{(k)} = \phi_P\left(\boldsymbol{V}_P^{(k)}\right)$$

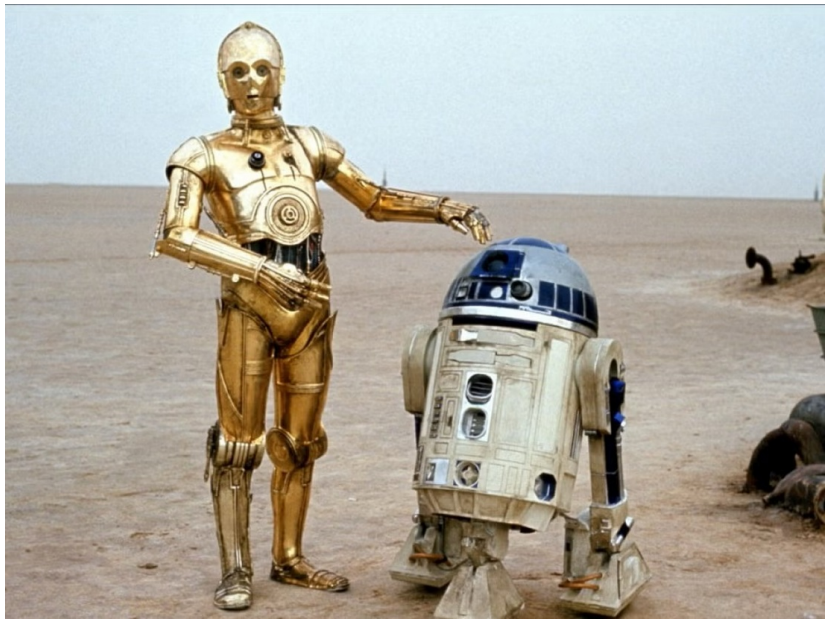
$$\boldsymbol{Y}_{PP}\left(\boldsymbol{V}_P^{(k+1)} - 1\right) = \boldsymbol{I}_P^{(k)}$$

(and its equivalent methods)

These are the two relevant AC Power Flow algorithms today.

Newton-Raphson is far more prevalent.

The Fixed-Point Current-Injection method is used by some software for distribution networks.



Power Flow Analysis – Part 4

ELEN90060

Keith Moffat

Bonus: Why is Y not invertible?

(no shunt admittance case)

$$Y = A^T \mathbf{Y}_{br} A$$

By definition of the incidence matrix, each row of A computes a voltage difference, so

$$A\mathbf{1} = 0$$

Therefore,

$$Y\mathbf{1} = A^T \mathbf{Y}_{br} A\mathbf{1} = 0$$

Thus, Y has a nonzero null vector is not invertible.

Physical meaning: Y cannot be inverted because there is ambiguity in the V, I relationship; Adding the same voltage to every bus does not change the current injections and thus the current injections do not uniquely define a voltage.

Bonus: Why is Y_{PP} invertible?

(no shunt admittance case)

$$Y_{PP} = A_P^T \boldsymbol{Y}_{br} A_P$$

For A_P to have full column rank, if $A_P x = 0$ then $x = 0$.
If $A_P x = 0$, $x_1 = 0$ because the first row of A_P only has an entry in the first column.
For a connected network, $A_P x = 0$ and the structure of A_P also means that all entries of x are equal.
Thus, if $A_P x = 0$, then $x = 0$ and thus A_P has full column rank.

Furthermore, the product of a full-row-rank matrix (A_P^T) by a positive-definite-real matrix ($\boldsymbol{Y}_{br}$) by a full-column-rank matrix (A_P) is full rank.

Thus, Y_{PP} is invertible.

Note: an alternative proof uses, for a connected network with no shunts, $nullspace(Y) = \{1\}$

Bonus: Linear Algebra Insight

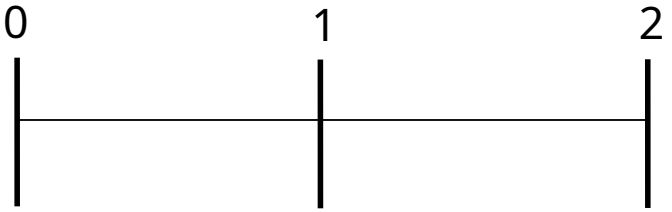
Before, when we talked about sparse matrix factorization to accelerate computation of linear equations, we briefly talked about LDL factorization. Radial networks are a rare case with a physical explanation of the LDL factorization

For radial networks:

A_P is square-lower-triangular

Y_{br} is diagonal

$Y_{PP} = A_P^T Y_{br} A_P$ *is an LDL (or UDU) factorization*



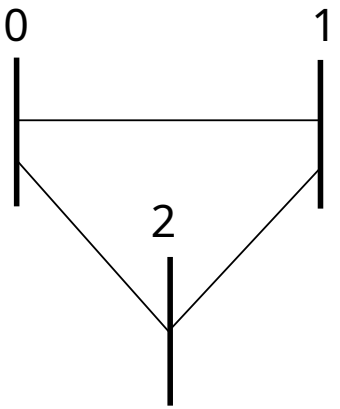
$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow A_P = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

For meshed (non-radial) networks:

A_P is *not* square-lower-triangular

Y_{br} is diagonal

$Y_{PP} = A_P^T Y_{br} A_P$ *is not an LDL (or UDU) factorization*



$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow A_P = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$$

(not square)