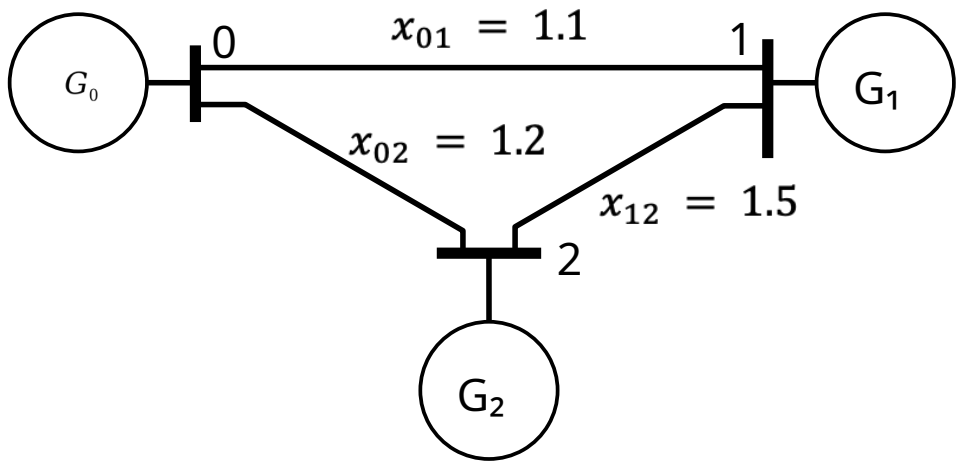
Power Flow Analysis – Review

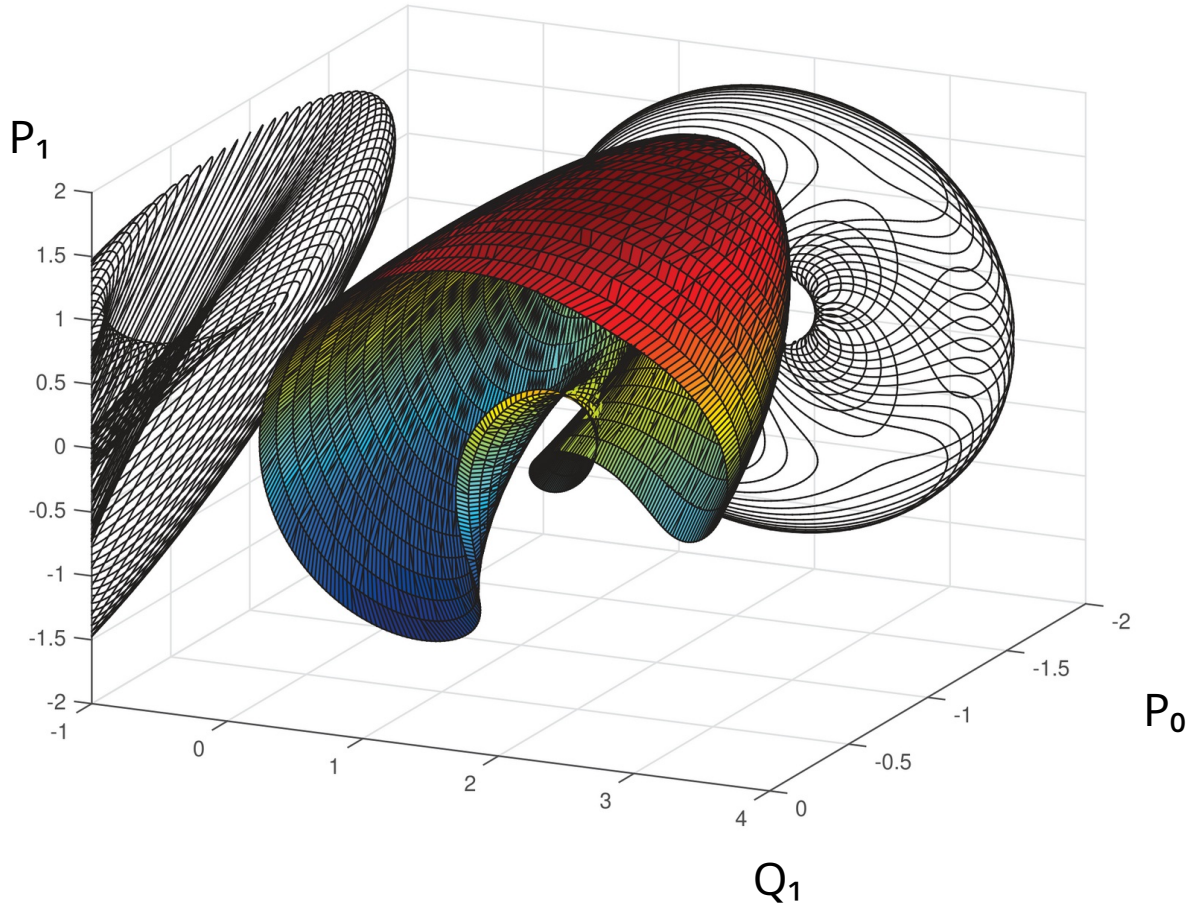
ELEN90060

Keith Moffat



$$V_0 = V_1 = V_2 = 1.05$$

Power flow manifold



https://doi.org/10.1109/59.932273
https://molzahn.github.io/pubs/molzahn_hiskens-fnt2019.pdf#page=23

Outline

1. Power System Notation: One-line diagrams
   → This is how power systems are commonly depicted

2. Power Flow in a Two-Bus System
   → Solvable. But real systems have thousands of buses

3. Power Flow Analysis
   - Nodal Analysis, Admittance Matrix
   - Power Flow Equations, Types of Buses
   - Implicit/Explicit Equations

Phasors

For fixed-frequency AC, a phasor keeps the amplitude and phase and hides the common time rotation.

Math

0. Starting point: the time domain signal

$$x(t) = A\cos(\omega t + \theta)$$

1. Embed the real sinusoid in a complex exponential

$$x(t) = \mathrm{Re}\{Ae^{j(\omega t+\theta)}\}$$

2. Factor out the known frequency ω

$$Ae^{j(\omega t+\theta)} = \left(Ae^{j\theta}\right)e^{j\omega t}$$

3. What remains is the phasor

$$Ae^{j\theta}$$

Power systems use the nominal 50/60 Hz frequency for steady-state phasors.

$$\mathbf{X} = Ae^{j\theta} \quad \Leftrightarrow \quad x(t) = \mathrm{Re}\{Xe^{j\omega t}\}$$

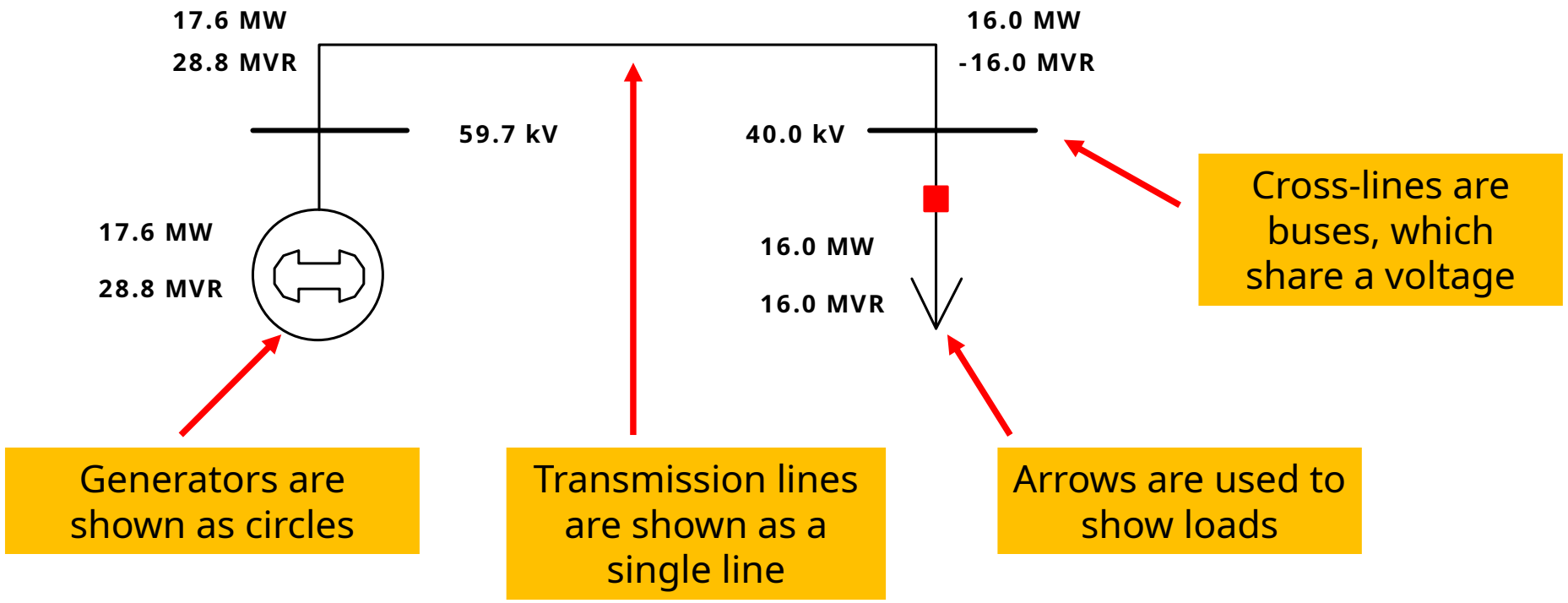
Payoff

Steady-state AC circuits become algebra:

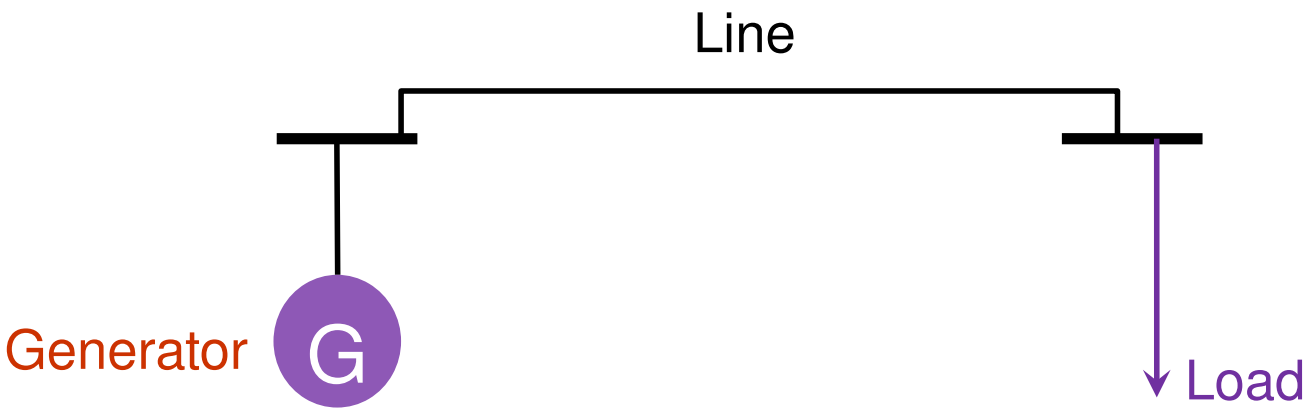
$$\mathbf{I} = \mathbf{YV}, \qquad \mathbf{S} = \mathbf{V}\bar{\boldsymbol{I}}$$

Power System Notation

Power system components are usually shown as “one-line diagrams.”



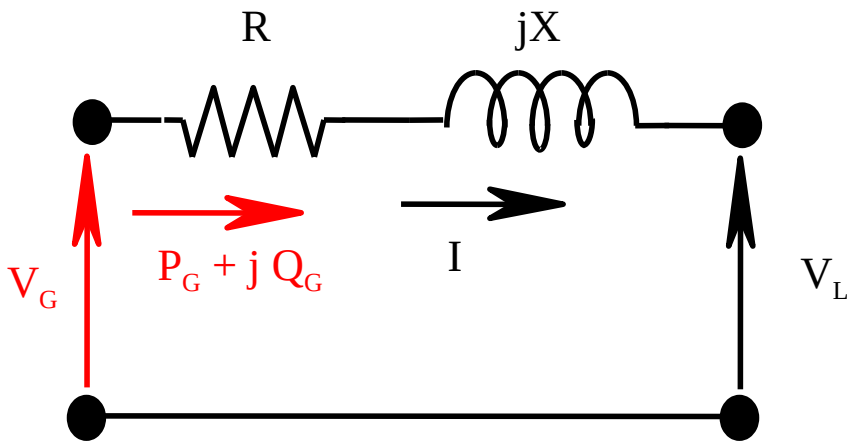
Power Flow in a Two-Bus System



Physical Laws

- Ohm’s Law
  - Current through a line → voltage across the line
- Kirchoff’s Current Law (KCL)
  - Current into a line or bus equals current out
- Conservation of Power
  - Voltage and current injection at a bus → power at that bus

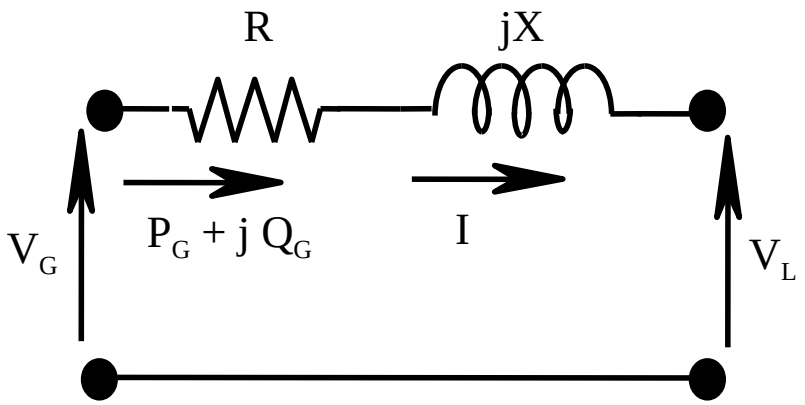
Power Flow in a Two-Bus System
Sending voltage and sending powers known 1/2



Neglect shunt admittance (susceptance) of the line.

Specify the voltage and the power injected at the **generator end** of the line.

Power Flow in a Two-Bus System
Sending voltage and sending powers known 2/2



$$\mathbf{S}_G = P_G + jQ_G = \mathbf{V}_G\,\bar{\mathbf{I}}$$

$$\mathbf{I} = \frac{P_G - jQ_G}{\bar{\mathbf{V}}_G}$$

$$\mathbf{V}_L = \mathbf{V}_G - (R + jX)\mathbf{I}$$

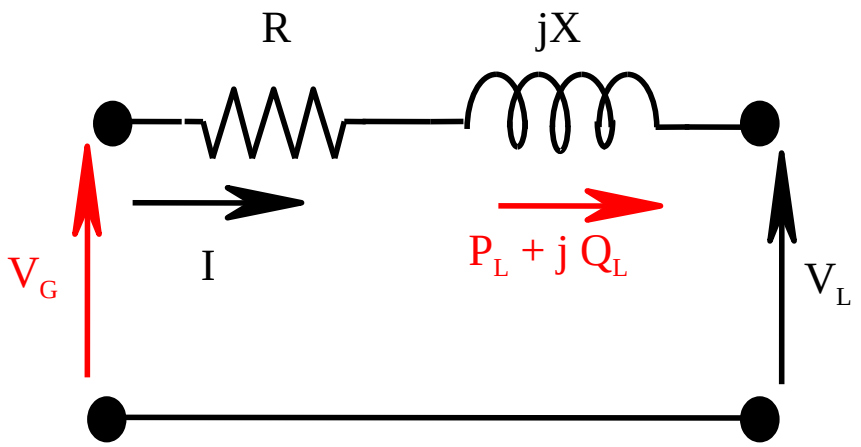
$$\mathbf{V}_L = \mathbf{V}_G - (R + jX)\frac{P_G - jQ_G}{\bar{\mathbf{V}}_G}$$

$$\mathbf{V}_G = \bar{\mathbf{V}}_G = V_G\angle 0° = V_G$$

$$\mathbf{V}_L = V_G - \frac{RP_G + XQ_G}{V_G} - j\frac{XP_G - RQ_G}{V_G}$$

Closed-form solution

Power Flow in a Two-Bus System
Sending voltage and receiving powers known 1/2



This is the typical power system problem.

- Voltage given at the **generator end**
- Load given at the **load end**
- Load voltage is unknown

Why do we not know P_G and Q_G?

Why do we not know I?

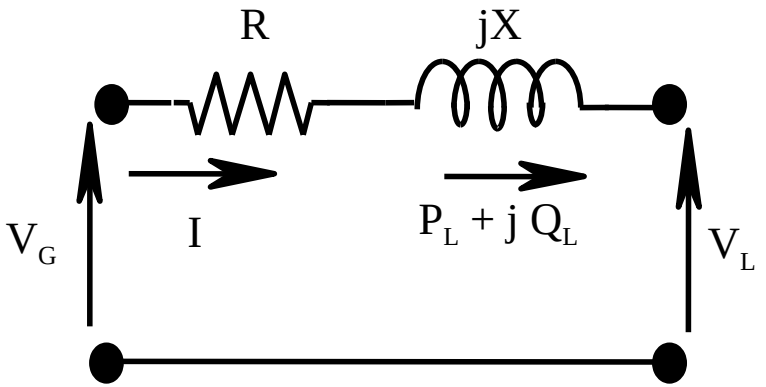
If we solve for the load voltage, the state is known (It's easy to calc. P_G and Q_G)

Power Flow in a Two-Bus System
Sending voltage and receiving powers known 2/2

$$\mathbf{S}_L = P_L + jQ_L = \mathbf{V}_L \bar{\mathbf{I}}$$

$$\mathbf{V}_L = \mathbf{V}_G - (R + jX)\mathbf{I}$$

$$\mathbf{V}_L = \mathbf{V}_G - (R + jX)\frac{P_L - jQ_L}{\bar{\mathbf{V}}_L}$$

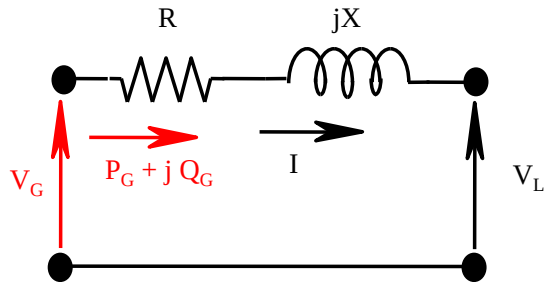


Where is the nonlinearity?

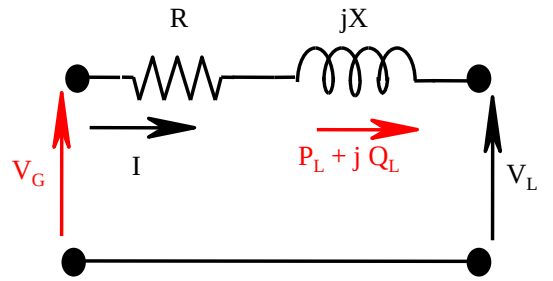
- **Non-linear equation**
- **No closed form solution**
- Solution requires an **iterative method**

Why are the two cases different?

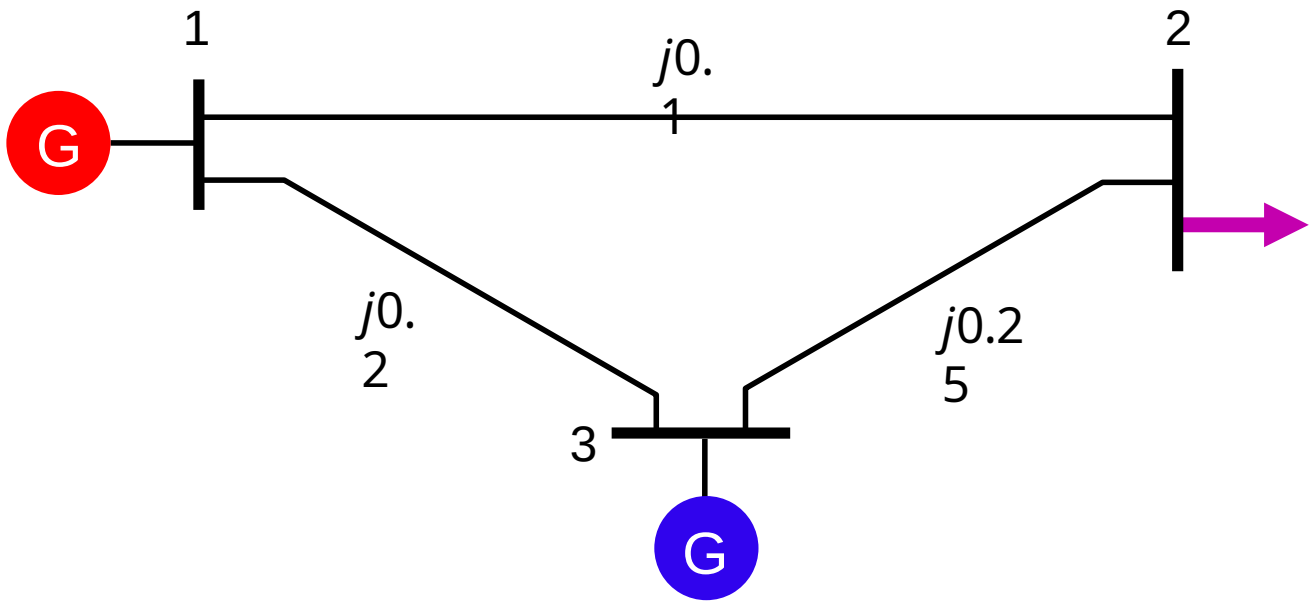
- *First Case*
  - Complex power and voltage are specified at the same bus, we therefore calculate the injected current



- *Second Case*
  - Complex power and voltage are specified at different buses
  - The **losses are unknown**
  - Thus, we do not know the current/power injection at bus G
  - **This is the typical power system problem**



A Three-Bus Example

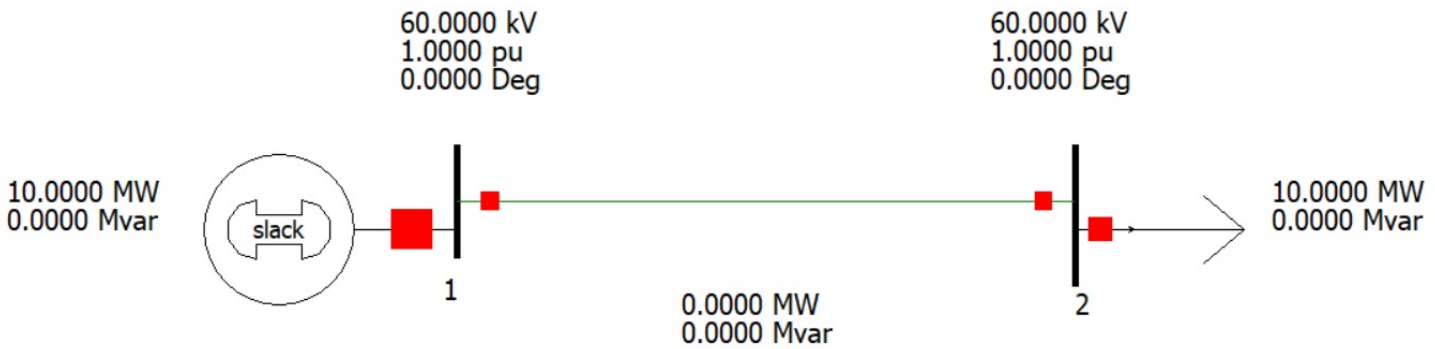


What about for this system?

Given the load, do we know the line flows?

PowerWorld Example

- A three-phase synchronous generator maintains the nominal voltage of 60 kV (line-to-line) at one end of a transmission line. A three-phase active power load of 10 MW is connected to the other end.
  - The line's inductive reactance per phase is equal to 40 Ω
  - Using per unit, calculate the L-L voltage of the load in kV



Physical Laws

- Ohm's Law
  - Current through a line → voltage across the line
- Kirchoff's Current Law (KCL)
  - Current into a line or bus equals current out
- Conservation of Power
  - Voltage and current injection at a bus → power at that bus

$$\boldsymbol{I = YV}$$

$$\boldsymbol{S = V\bar{I}}$$

Nodal Analysis Method

1. Build the admittance matrix
2. Combine $\boldsymbol{I} = \boldsymbol{YV}$ and $\boldsymbol{S} = \boldsymbol{V}\bar{\boldsymbol{I}}$ into a set of nonlinear equations
3. Solve the set of nonlinear equations

- The nodal analysis method can be applied to AC or DC circuits
  - AC → complex variables (phasors)
  - DC → real-valued variables

Both are nonlinear

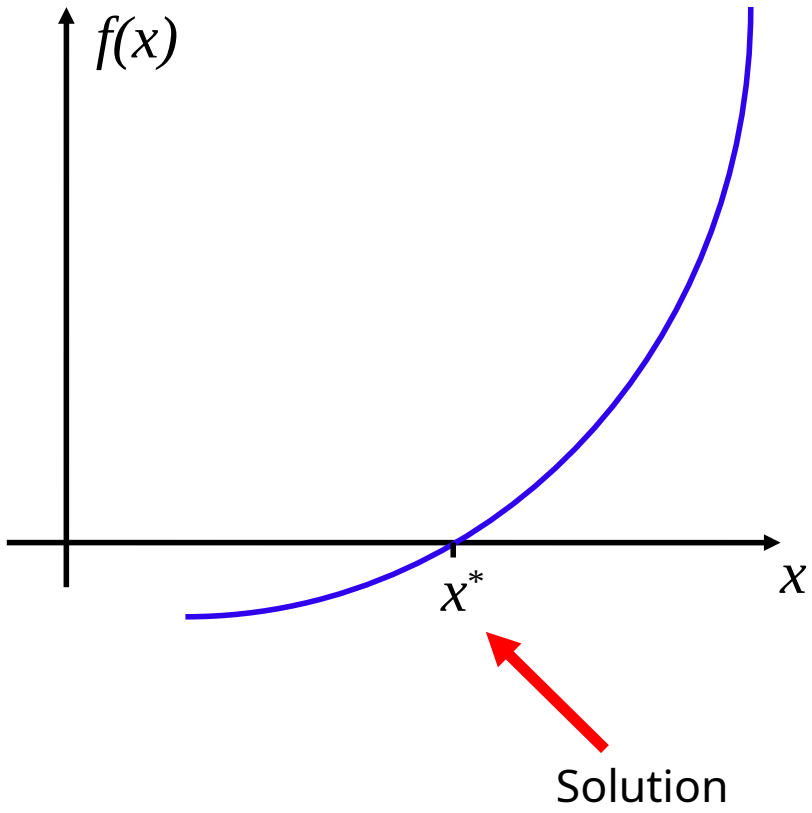
What is the source of nonlinearity?

- Solving sets of nonlinear equations is hard
  - We have multiple algorithms to do this
  - Newton-Raphson is the most popular algorithm

Are nonlinear systems always solvable?

Solving nonlinear equations

- Implicit equations: set of non-linear equations
- No direct solution
- Need an **iterative** solution method

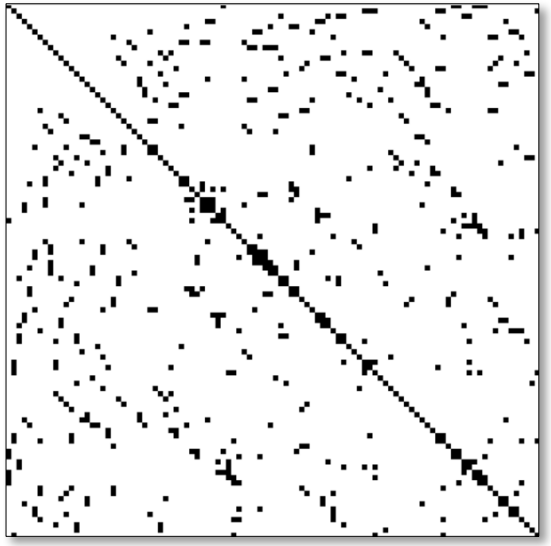


Building the Admittance Matrix Y

- Build the admittance matrix directly:
  - Diagonal element (i , i):
    - sum of the admittances connected to node I, including the shunt admittance
  - Off-diagonal element (i , j):
    - minus the admittance of the branch between nodes i and j
  - No branch between nodes i and j → element (i , j) is zero

Admittance Matrix Y | Some Notes

- Elements of Y depend only on admittances of branches of the network
- If there is no branch between nodes i and j, element Y_{ij} is zero
- For a large network such as a power system, most of the elements of Y are zero
- Y is a sparse matrix



Sparse → Opportunities for computational tricks

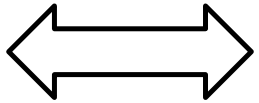
Two Representations of AC Power Flow

Real-valued bus injection equations

$$P_k = \sum_{i \in N_k} V_k V_i [G_{ki} \cos\theta_{ki} + B_{ki} \sin\theta_{ki}]$$

$$Q_k = \sum_{i \in N_k} V_k V_i [G_{ki} \sin\theta_{ki} - B_{ki} \cos\theta_{ki}]$$

$$\theta_{ki} = \theta_k - \theta_i$$



Complex-valued power equations

$$\boldsymbol{S} = \mathrm{diag}(\boldsymbol{V})\bar{\boldsymbol{I}}, \qquad \boldsymbol{I} = \boldsymbol{Y}\,\boldsymbol{V}$$

$$\boldsymbol{S} = \mathrm{diag}(\boldsymbol{V})\overline{\boldsymbol{Y}\boldsymbol{V}}$$

Set of non-linear equations that express the balance of active and reactive power at each bus

Types of Buses

- Active and reactive power balance must hold for the whole system

  (sum of all power injections) + (losses on all lines) = 0

If we don’t know the line flows, do we know the losses?

Can we assert P and Q injections for every bus?

Types of Buses

Active and reactive power balance must hold for the whole system

(sum of all power injections) + (losses on all lines) = 0

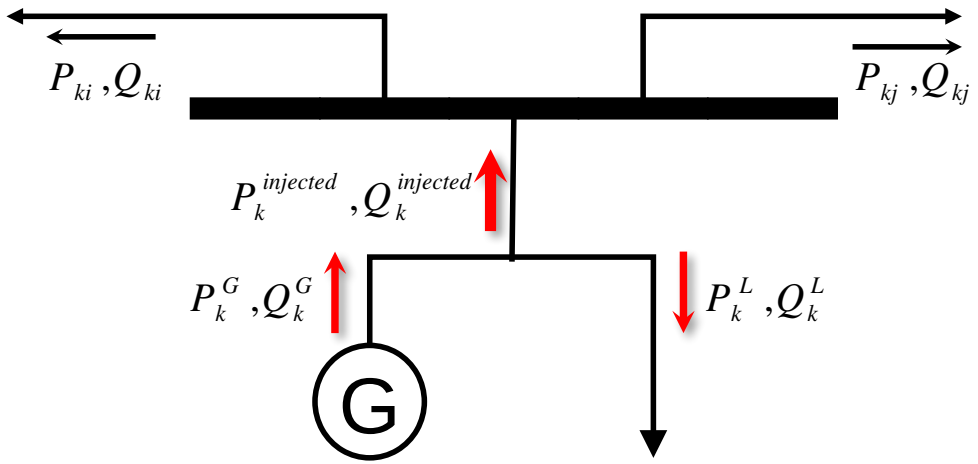
If we don't know the line flows, do we know the losses?

Can we assert P and Q injections for every bus?

Three types of buses:

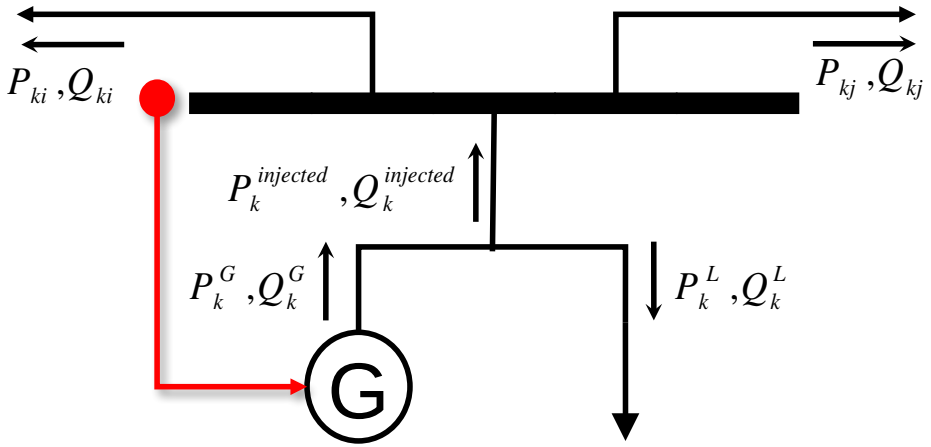
- **PQ bus**
- **PV bus**
- **Slack bus** (aka swing bus)

Types of Buses | PQ bus



- At a PQ bus
  - $P^{injected}$ is specified
  - $Q^{injected}$ is specified
  - Voltage magnitude and angle are unknown
  - Voltage magnitude and angle are such that the active and reactive flows in the lines connected to the PQ bus match the injections
- Load buses are usually PQ buses

Types of Buses | PV bus

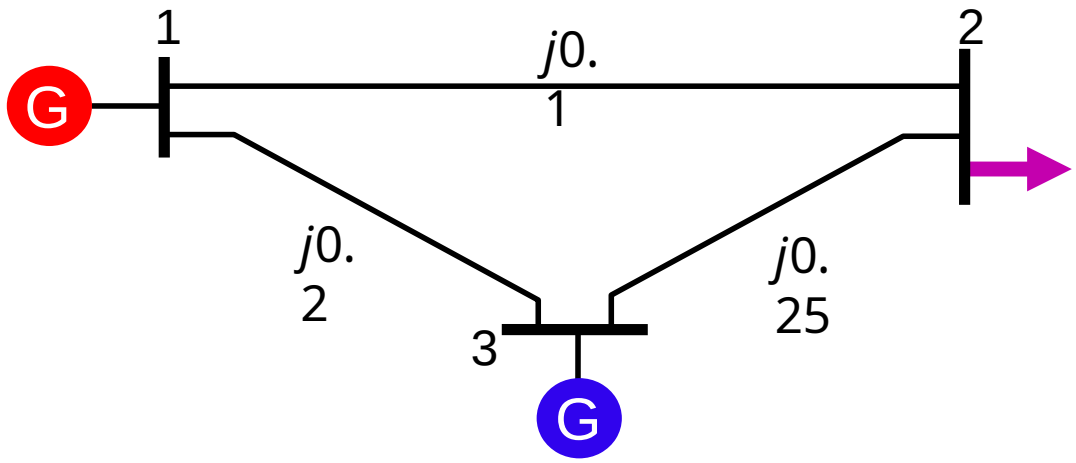


- At a PV bus
  - $P^{injected}$ is specified
  - Voltage magnitude is specified
  - Voltage angle is unknown
  - $Q^{injected}$ is unknown
- $Q^{injected}$ adjusted to keep voltage magnitude constant
- Must have a variable var source at each PV bus
- Most generation buses are PV buses (in transmission)

Balancing the Power Losses | The Slack Bus

- **Power balance solution**: leave P^{injected} unspecified at one bus: the slack bus
- During the power flow calculation, $P_{\text{slack}}^{\text{injected}}$ is adjusted to keep the system in balance
- Also, need reference for the angles
  - set voltage angle at slack bus arbitrarily (equal to 0°)
- P, Q unspecified
- Voltage magnitude and angle are specified at the slack
- A generator bus is usually chosen as slack bus because the active and reactive power of that generator can be adjusted by the power flow

Three Bus Example

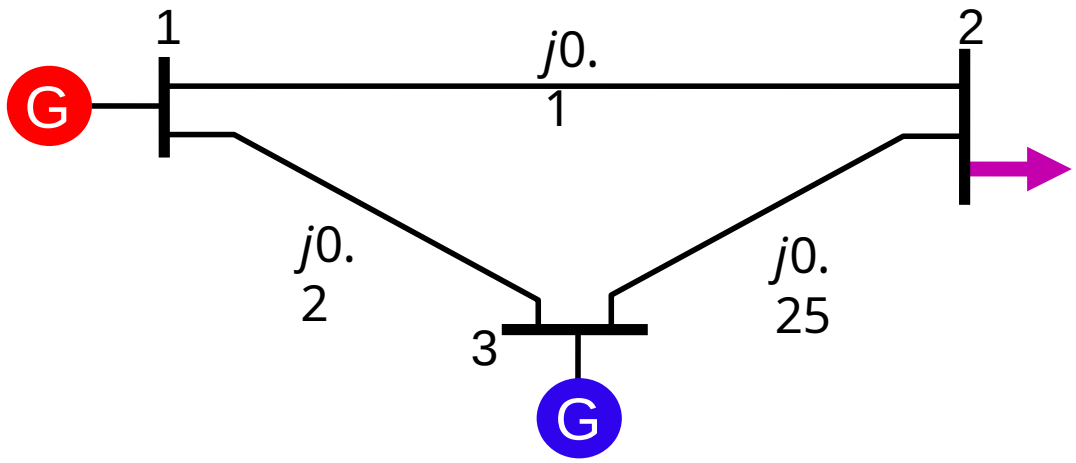


All values are per unit **impedances**

Write the power flow equations for this system using the following steps.

Work in groups: Step 1) write the Ybus matrix

Three Bus Example | Y matrix



Calculate the branch/line admittances, then the Y matrix

$$Y = G + jB = \begin{bmatrix} -j15 & j10 & j5 \\ j10 & -j14 & j4 \\ j5 & j4 & -j9 \end{bmatrix}$$

Three-Bus Example | Power Flow Equations

$$P_k^{injected} = \sum_{i \in N_k} V_k V_i [G_{ki} \cos\theta_{ki} + B_{ki} \sin\theta_{ki}]$$

$$Q_k^{injected} = \sum V_k V_i [G_{ki} \sin\theta_{ki} - B_{ki} \cos\theta_{ki}]$$

- Two equations for each node
  - First equation: balance of active power at that node
  - Second equation: balance of reactive power at that node
- Left hand side: power injected at the node (generator or load)
- Right hand side: power flowing in the branches

Work in groups: Step 2) write the power flow equations

A Three-Bus Example | Power Flow Equations

$$P_k^{injected} = \sum_{i \in N_k} V_k V_i [G_{ki} \cos\theta_{ki} + B_{ki} \sin\theta_{ki}]$$
$$Q_k^{injected} = \sum V_k V_i [G_{ki} \sin\theta_{ki} - B_{ki} \cos\theta_{ki}]$$

$$Y = G + jB = \begin{bmatrix} -j15 & j10 & j5 \\ j10 & -j14 & j4 \\ j5 & j4 & -j9 \end{bmatrix}$$

$$\begin{aligned}
P_1 &= -15V_1V_1 \sin\theta_{11} + 10V_1V_2 \sin\theta_{12} + 5V_1V_3 \sin\theta_{13} \\
Q_1 &= +15V_1V_1 \cos\theta_{11} - 10V_1V_2 \cos\theta_{12} - 5V_1V_3 \cos\theta_{13} \\
P_2 &= +10V_2V_1 \sin\theta_{21} - 14V_2V_2 \sin\theta_{22} + 4V_2V_3 \sin\theta_{23} \\
Q_2 &= -10V_2V_1 \cos\theta_{21} + 14V_2V_2 \cos\theta_{22} - 4V_2V_3 \cos\theta_{23} \\
P_3 &= +5V_3V_1 \sin\theta_{31} + 4V_3V_2 \sin\theta_{32} - 9V_3V_3 \sin\theta_{33} \\
Q_3 &= -5V_3V_1 \cos\theta_{31} - 4V_3V_2 \cos\theta_{32} + 9V_3V_3 \cos\theta_{33}
\end{aligned}$$

Note: all the “G” terms (conductances) are zero because we neglected the resistance of the lines

Three-Bus Example | Power Flow Equations

- Since we have:

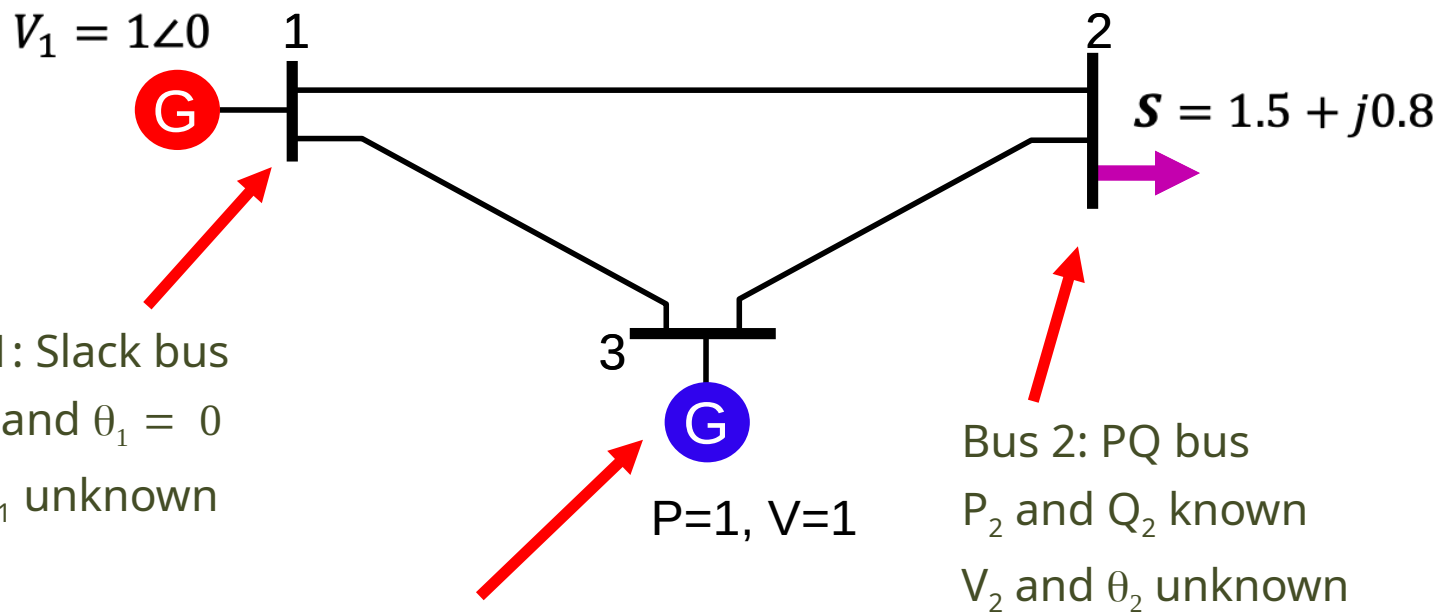
$$\sin\theta_{11} = \sin\theta_{22} = \sin\theta_{33} = \sin 0^\circ = 0$$
$$\cos\theta_{11} = \cos\theta_{22} = \cos\theta_{33} = \cos 0^\circ = 1$$

- The power flow equations become:

$$P_1 = +10V_1V_2\sin\theta_{12} + 5V_1V_3\sin\theta_{13}$$
$$Q_1 = +15V_1^2 - 10V_1V_2\cos\theta_{12} - 5V_1V_3\cos\theta_{13}$$
$$P_2 = +10V_2V_1\sin\theta_{21} + 4V_2V_3\sin\theta_{23}$$
$$Q_2 = -10V_2V_1\cos\theta_{21} + 14V_2^2 - 4V_2V_3\cos\theta_{23}$$
$$P_3 = +5V_3V_1\sin\theta_{31} + 4V_3V_2\sin\theta_{32}$$
$$Q_3 = -5V_3V_1\cos\theta_{31} - 4V_3V_2\cos\theta_{32} + 9V_3^2$$

6 equations

Three-Bus Example | Bus Types



Bus 1: Slack bus

V_1=1 and $\theta_1 = 0$

P_1, Q_1 unknown

Bus 3: PV bus

P_3 and V_3 known

Q_3 and θ_3 unknown

Bus 2: PQ bus

P_2 and Q_2 known

V_2 and θ_2 unknown

6 unknowns

Three-Bus Example | Separating the Equations

Implicit equations: known power injections

$$P_2 = 10V_2V_1 \sin\theta_{21} + 4V_2V_3 \sin\theta_{23}$$
$$P_3 = 5V_3V_1 \sin\theta_{31} + 4V_3V_2 \sin\theta_{32}$$
$$Q_2 = -10V_2V_1 \cos\theta_{21} + 14V_2^2 - 4V_2V_3 \cos\theta_{23}$$

3 equations
3 unknowns

solve these

then solve these

Explicit equations: unknown power injections

$$P_1 = 10V_1V_2 \sin\theta_{12} + 5V_1V_3 \sin\theta_{13}$$
$$Q_1 = 15V_1^2 - 10V_1V_2 \cos\theta_{12} - 5V_1V_3 \cos\theta_{13}$$
$$Q_3 = -5V_3V_1 \cos\theta_{31} - 4V_3V_2 \cos\theta_{32} + 9V_3^2$$

3 equations
3 unknowns

A Three-Bus Example | Implicit Equations

- Known variables: $P_2, Q_2, P_3, V_1, V_3, \theta_1$
- Unknown variables: V_2, θ_2, θ_3

P_2 equation: $-1.5 = 10V_2 \sin\theta_{21} + 4V_2 \sin\theta_{23}$

P_3 equation: $1 = 5 \sin\theta_{31} + 4V_2 \sin\theta_{32}$

Q_2 equation: $-0.8 = -10V_2 \cos\theta_{21} + 14V_2^2 - 4V_2 \cos\theta_{23}$

A Three-Bus Example | Explicit Equations

- Known variables: V_1, V_3, θ_1
- Solution of implicit equations: V_2, θ_2, θ_3
- Unknown variables: P_1, Q_1, Q_3

$$P_1 = 10V_2 \sin\theta_{12} + 5\sin\theta_{13}$$
$$Q_1 = 15 - 10V_2 \cos\theta_{12} - 5\cos\theta_{13}$$
$$Q_3 = -5\cos\theta_{31} - 4V_2 \cos\theta_{32} + 9$$

Power Flow Analysis – Review

ELEN90060

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